Relationship between the inverse scattering techniques of Belinskii–Zakharov and Hauser–Ernst in general relativity [a)]

Christopher M. Cosgrove [b)]
W. K. Kellogg Radiation Laboratory, California Institute of Technology, Pasadena, California 91125

(Received 22 July 1981; accepted for publication 16 October 1981)

We make a quantitative comparison between the *pure-nonsoliton* part of the inverse scattering method of Belinskii and Zakharov (BZ) and the homogeneous Hilbert problem of Hauser and Ernst (HE), these being two independent representations of an infinite-dimensional subgroup $\mathscr{K}$ of the Geroch group **K** of invariance transformations for spacetimes with two commuting Killing vectors. An explicit formula for the BZ representing matrix function $G_0(\lambda)$ in terms of the HE representing matrix function $u(t)$ is derived. It is shown how certain solution generating techniques (e.g., Harrison's Bäcklund transformation, HKX transformation, generation of Weyl solution from flat space, generation of n-Kerr–NUT solution from n-Schwarzschild) can be derived directly from the BZ formalism, including the soliton part in some cases, thereby bringing our understanding of the BZ formalism up to the level of the more fully developed HE formalism. A technical point which needed to be resolved along the way was how to analytically continue the complex matrix potential $F(t)$ across a quadratic branch cut and onto the second Riemann sheet. Finally, we consider how the subgroup $\mathscr{K} \subset \mathbf{K}$ represented by the BZ and HE formalisms can be enlarged either by simple limiting transitions or by relaxing boundary conditions.

PACS numbers: 04.20.Cv, 02.30. + g

1. INTRODUCTION AND PRELIMINARY

The "inverse scattering" technique is an attractive and powerful formalism for solving certain systems of nonlinear partial differential equations of mathematical physics. The essential feature of the method is to write down an overdetermined system of linear eigenvalue equations (a so-called L–A pair) whose integrability conditions are the given nonlinear system. Then methods of functional analysis can be applied to generate new solutions of the linear system from old, and hence new solutions of the original system from old. Most if not all of the partial differential equations which are known to yield to the inverse scattering method also exhibit other remarkable phenomena such as an infinity of conservation laws, Bäcklund transformations, Bianchi diagrams, and nonlinear superposition principles.[1]

In the case of Einstein's gravitational field equations with two commuting Killing vectors, there are two rival inverse scattering methods found independently by Belinskii and Zakharov[2] (BZ) and Hauser and Ernst[3–5] (HE). The latter provide an explicit representation of the infinite-dimensional Geroch group[6] **K** of invariance transformations (strictly speaking, a large subgroup $\mathscr{K}_L$ of **K**[5]) and was derived initially from the earlier representation of the associated Lie algebra given by Kinnersley and Chitre[7–10] (KC). The HE formalism has been developed and applied in a series of papers,[3–5,11] some of the highlights of which are the following: reduction of certain ingenious solution-generation techniques of Kinnersley and coworkers (e.g., flat space→Weyl,[9] Schwarzschild→Kerr,[10] HKX transformation[12]) to relatively simple problems in complex-variable theory[3,4]; a proof of Geroch's conjecture[6] that the **K** group is (multiply) transitive on the space of solutions in a certain well-defined sense[5]; and recognition that Harrison's Bäcklund transformation[13,14] is contained in the **K** group.[11] Belinskii and Zakharov[2] separate invariance transformations into "soliton" and "nonsoliton" parts, these ideas being taken from the theory of certain nonlinear wave equations (or their elliptic counterparts) in quantum physics.[1] Subsequent papers[15] have concentrated almost exclusively on the *pure-soliton* BZ transformations which are permutable Bäcklund transformations and have been shown to be closely related to the Harrison and HKX transformations.[16]

It is apparent at a glance that the HE formalism and the *pure-nonsoliton* part of the BZ formalism are qualitatively similar: both reduce the nonlinear field equations to an homogeneous Hilbert problem (HHP) or regular Riemann problem in complex functional analysis. However, the problem of providing quantitative mathematical formulas connecting the two formalisms is far from straightforward even though the exact relation between the respective eigenfunctions $F(t)$ and $\psi(\lambda)$ is known[16] (see Appendix B). What is needed is an explicit formula for the BZ representing matrix function $G_0(\lambda)$ ("scattering data") in terms of the HE representing matrix function $u(t)$. The result [Eq. (2.35) below] shows how a particular transformation in one formalism can be written in the other.

In Sec. 3, we show how to solve the BZ homogeneous Hilbert problem for four special classes of $G_0(\lambda)$ matrices. They are (A) Harrison's Bäcklund transformation,[13,14] (B) the null generalized HKX transformation,[12,16] (C) generation of the general Weyl or Einstein–Rosen solution from flat space,[9] and (D) **B**-group[10] generation of the nonlinear superposition of n Kerr–NUT particles from n Schwarzschild particles. The first three examples can be directly com-

[a)]Supported in part by the National Science Foundation [AST79-22012-A1].
[b)]Richard Chace Tolman Research Fellow.

pared with the corresponding calculations in the HE formalism. The fourth case, however, is quite different and belongs in the pure-soliton framework.

Before the explicit relation (2.35) connecting $G_0(\lambda)$ to $u(t)$ can be constructed, it will be necessary to make precise the assumptions about gauge, analyticity properties, and boundary conditions used in the BZ formalism. In particular, the boundary condition, $G_0(\infty) = I$ (unit 2×2 matrix) in Ref. 2 will be dropped as it is unnecessarily restrictive, and the requirement of symmetry of $G_0(\lambda)$ will be replaced by a more complicated symmetry condition in order to achieve internal consistency. The chosen boundary conditions, etc., were motivated by the corresponding conditions chosen by Hauser and Ernst. Hauser and Ernst have also studied the analyticity properties of the $F(t)$ potential in the complex t plane.[4,5] In order to understand the analyticity properties of the corresponding BZ eigenfunction $\psi(\lambda)$ in the complex λ plane, we need the exact form of the analytic continuation of $F(t)$ across a quadratic branch cut in the t plane and onto the second Riemann sheet. This problem is solved in Appendix A using some powerful ideas in Ref. 5.

With the boundary conditions and other assumptions in Sec. 2, the solutions of the HHP's for given $u(t)$ or $G_0(\lambda)$ are guaranteed to be unique. However, some of these conditions can be relaxed with the result that the solutions are no longer unique, but may depend on some arbitrary "constants" which are actually arbitrary functions of the spacetime coordinates. One must then substitute the class of solutions admitted by the HHP's directly into Einstein's equations and/or the equivalent eigenvalue equations in order to see which are genuine gravitational solutions (the final solutions may be unique or contain some integration constants). The mixed soliton and nonsoliton transformations of Belinskii and Zakharov[2] are an example of this phenomenon where the functions $\chi_1(\lambda)$ and $\chi_2(\lambda)$ or their inverses are allowed to have poles in the regions of the λ plane where they would be required to be analytic in the case of a pure-nonsoliton transformation. It is not clear, when the rules have been so changed, whether the pure-soliton and mixed transformations are in **K** itself or some larger group.[17] In fact, the $2n$-soliton transformation is necessarily in **K** because it is identical (up to some gauge and trivial transformations) to the $2n$-fold Harrison transformation[16] which is known to be in **K**.[11] In Ref. 16, we also proved that the $(2n-1)$-soliton transformation is the product of $2n-1$ Harrison transformations and the Kramer–Neugebauer involution[18] in any order. In Sec. 4, we discuss the possibility of including the Kramer–Neugebauer involution in a suitable analytic continuation of the subgroup $\mathscr{K}_L \subset \mathbf{K}$ represented by the HE formalism.

In Sec. 4, we also discuss more straightforward analytic continuations of $\mathscr{K}_L$ which can be represented in the HE formalism by relaxing boundary conditions at $t=\infty$. A simple example is the $s=\infty$ limit of the null (and nonnull) HKX transformation which was calculated in Ref. 16 by exponentiating the appropriate limiting infinitesimal transformation. In the HE formalism, the null HKX transformation with finite s is represented by a $u(t)$ matrix with a simple pole at $t=s$ *inside* the closed contour L. The limiting HKX transformation can be derived from an HHP with $u(t)=I$ and $X_-(t) = O(t)$ as $t\to\infty$, and an additional parameter can be incorporated by also allowing $u(t) = O(t)$ as $t\to\infty$. As in the case of the BZ soliton transformations, the solution of this modified HHP involves an arbitrary matrix function of the space-time coordinates which must be determined by substitution into Einstein's equations. By taking limits of more complicated transformations as singularities of $u(t)$ go to infinity, it is clear that $u(t)$ and $X_-(t)$ can be allowed to have virtually any type of singularity at $t=\infty$, including branch points and nonisolated essential singularities. The special boundary conditions at $t=\infty$ on $u(t)$ and $X_-(t)$ chosen by Hauser and Ernst are thus seen to be not necessary for the preservation of Einstein's equations but have the convenient property of guaranteeing uniqueness of the solution of the HHP for given $u(t)$ and input $F(t)$.

The gravitational field equations in the presence of two commuting Killing vectors, which are reducible to the Ernst equation[19] for a complex potential $\mathscr{E}$, are applicable to many problems both within and outside the general theory of relativity. For example, they apply to stationary axisymmetric vacuum fields,[20] cylindrical wave fields,[21] the interaction region of colliding plane waves,[22] electrostatic and magnetostatic Einstein–Maxwell fields,[23] classical nonabelian gauge theories,[24] self-dual Yang–Mills fields,[25] and are closely related to the principal chiral fields.[26] The largest body of literature applies to the case of stationary axisymmetric vacuum gravitational fields, but most of the results in one theory can be shared with the others.[27] In this paper, we wish to treat the stationary axisymmetric and cylindical wave cases together, keeping in mind that there is a simple complex coordinate transformation that maps one case onto the other. Also, many of the references already cited (e.g., some of Refs. 3, 4, 7–9, 11, 15) consider extensions of the theory of the **K** group or solitonic methods to stationary axisymmetric electrovac Einstein–Maxwell fields, but only the vacuum case will be treated in the present paper.

The canonical form of the metric of stationary axisymmetric space-time is[7,20]

$$ds^2 = f_{AB}dx^A dx^B - e^{2\Gamma}(d\rho^2 + dz^2), \tag{1.1}$$

$A, B = 1, 2, f_{11}f_{22} - (f_{12})^2 = -\rho^2$, where x^1, x^2, ρ, and z are time, azimuthal, radial, and axial coordinates, respectively, and f_{AB} and Γ are functions of ρ and z only. A change of variable,

$$\alpha = i\rho, \quad \beta = z, \tag{1.2}$$

(and change of sign of ds^2 to preserve signature -2) puts the metric in the form appropriate to cylindrically symmetric gravitational waves:

$$ds^2 = -f_{AB}dx^A dx^B + e^{2\Gamma}(-d\alpha^2 + d\beta^2). \tag{1.3}$$

Here, $f_{11}f_{22} - (f_{12})^2 = \alpha^2$, f_{AB} and Γ are functions of α and β only, and x^1, x^2, α, and β are axial, azimuthal, radial, and time coordinates, respectively. The coordinates α and β are to be identified with the fields $\alpha(\zeta,\eta)$ and $\beta(\zeta,\eta)$ of Ref. 2. When the BZ formalism is under consideration, we shall use the metric form (1.3) exclusively (Sec. 3D excepted), whereas in the HE formalism, we shall use (1.1) and (1.3) interchangeably.[27] The function Γ, which can be calculated from f_{AB} by quadrature, will play no further role in this paper.

When a particular solution of the elliptic (stationary case) or hyperbolic (wave case) field equations governing f_{AB} is given, a number of other important auxiliary potentials can be calculated. We refer the reader to the literature, especially Refs. 3–5, 7–10, and 16, for the field equations and defining relations for potentials and Refs. 2, 13–16, and 28 for pseudopotentials and eigenfunctions. We use the matrix notation,

$$\mathbf{g} = (f_{AB}) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} f & -f\omega \\ -f\omega & f\omega^2 - \rho^2 f^{-1} \end{pmatrix}, \tag{1.4}$$

and so $\mathbf{g}$ is symmetric and

$$\det \mathbf{g} = -\rho^2 \quad \text{or} \quad \alpha^2. \tag{1.5}$$

The SL(2)-tensor notation used extensively in Refs. 7–10 and 16 will not be used here except for brief appearances in Secs. 3B and 4. From the real matrix potential $\mathbf{g}$, one can construct a complex matrix potential,[7]

$$H = (H_{AB}) = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \tag{1.6}$$

[see Eq. (A2)] for which

$$\mathbf{g} = \operatorname{Re} H \quad \text{or} \quad f_{AB} = \operatorname{Re} H_{AB}, \tag{1.7}$$

and $H_{12} - H_{21} = 2iz$ or $2i\beta$. The 11 component of H is the complex Ernst potential,[19]

$$\mathscr{E} = H_{11} = f + i\psi. \tag{1.8}$$

The inverse scattering technique is applicable to this problem because it is possible to construct a complex matrix potential ("eigenfunction") $F(t)$[3–5,9–12,16,29] which is a nontrivial function of *three* variables, (ρ,z,t) or (α,β,t), by solving a simple linear partial differential equation, Eq. (A1) (or nonlinear characteristic equation[10] or Riccati equation[16]). This was originally introduced in Ref. 9 as a generating function of potentials,

$$F(t) = i\epsilon + tH + t^2 H^{(2)} + t^3 H^{(3)} + \cdots \tag{1.9}$$

[cf. Eqs. (A6a,b)], where

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.10}$$

Hauser and Ernst used $F(t)$ as their basic field variable and determined some of its analyticity properties in the complex t plane (t may be called a "spectral parameter.") There is the important result:

Theorem (Hauser and Ernst[5]): In a (ρ,z) domain containing the origin (0,0) for which $\mathscr{E}$ is an analytic function of (ρ,z) and $f \neq 0$, there is a unique gauge for $F(t)$ such that $F(t)$ is analytic in the whole complex t plane and

$$F(t) \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \text{ is analytic at } t = \infty, \tag{1.11}$$

except for two quadratic branch points of index $-\frac{1}{2}$, joined by a cut, at the zeros of

$$S(t) = [(1 - 2tz)^2 + 4t^2\rho^2]^{1/2}, \quad S(0) = 1. \tag{1.12}$$

We shall call this gauge "special HE gauge" as in Ref. 11. A change of gauge [see Eqs. (A5) and (A8)] introduces (ρ,z)-independent singularities in the finite t plane and/or at $t = \infty$.[30] If $\mathscr{E}$ is analytic and $f \neq 0$ at a point $(0,z_0)$ of the z axis, but not at (0,0), we assume that the regular point has been brought to the origin by a translation, $z \to z - z_0$. If $\mathscr{E}$ is singular and/or $f = 0$ along the whole z axis, then the minimally singular form of $F(t)$ may be quite different [see Eq. (4.27) below and Ref. 4]. Analytic continuation of $F(t)$ in special HE gauge across the branch cut and onto the second Riemann sheet will reveal (ρ,z)-independent singularities at $t = 0$ and in places that depend on the functional form of $\mathscr{E}(0,z)$. The exact functional relation between the two values of $F(t)$ on the two Riemann sheets [we write $\widetilde{F}(t)$ for the second sheet] is calculated in Appendix A.

Belinskii and Zakharov use a different matrix potential, $\psi(\lambda) = \psi(\alpha,\beta,\lambda)$, and different spectral parameter λ whose relations to $F(t)$ and t, respectively, are discussed in Ref. 16 and Appendix B here. Note that the complex λ plane maps onto the double-sheeted Riemann t surface which is the domain of the function,

$$S(t) = [(1 - 2t\beta)^2 - 4t^2\alpha^2]^{1/2}, \quad S(0) = 1. \tag{1.12$'$}$$

We restrict attention to (α,β) domains for which $|\beta| > \alpha > 0$ so that the image of the circle $|\lambda| = \alpha$ is the finite line segment along the real axis of the t plane joining the zeros of $S(t)$ traced in both directions and not passing through $t = 0$. The λ-values, λ and α^2/λ, map to the same t-values, but on different Riemann sheets. Appendix B gives the exact functional relation between $\psi(\lambda)$ and $\psi(\alpha^2/\lambda)$.

Finally, we wish to discuss briefly the homogeneous Hilbert problem (HHP) posed by Hauser and Ernst to provide simultaneously an explicit representation of a large subgroup ($\mathscr{K}_L$ of Ref. 5) of the Geroch group $\mathbf{K}$ and a practical method of generating new solutions from old. The reader is referred to Refs. 3–5 for a more detailed treatment, and Ref. 11 for notations and conventions.[29]

The transformation group $\mathscr{K}_L \subset \mathbf{K}$ is isomorphic to the Lie group of 2×2 matrix functions $u(t)$ of a complex variable t, subject to the conditions:

$$\det u(t) = 1; \tag{1.13a}$$

$$u(t) \quad \text{is real for real } t; \tag{1.13b}$$

$$u(t) \quad \text{is analytic in a neighborhood of } t = \infty; \tag{1.13c}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} u(t) \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \quad \text{is analytic at } t = \infty. \tag{1.13d}$$

The group product is the usual matrix product. Next, draw a simple closed contour L in the complex t plane enclosing all the singularities of $u(t)$ and symmetric about the real axis. The interior of L is denoted L_+ and the exterior L_-. Then choose a particular gravitational solution f_{AB} or equivalent solution $\mathscr{E}$ of the Ernst equation[19] and calculate the $F(t)$ potential *in special HE gauge*. Restrict the (ρ,z) domain to a region sufficiently close to (0,0) so that the two quadratic branch points of $F(t)$ and the cut joining them lie wholly in L_-. So $F(t)$ is analytic in $L + L_+$ and $u(t)$ is analytic in $L + L_-$.

A new solution $\mathscr{E}'$, together with associated potential $F'(t)$ in special HE gauge, which is the transform of $\mathscr{E}$ under the element of $\mathscr{K}_L$ represented by $u(t)$ can be calculated by solving the following homogeneous Hilbert problem: determine a matrix function $X_-(t)$ analytic in $L + L_-$ and a matrix function $X_+(t)$ or, equivalently, $F'(t) = X_+(t)F(t)$, analytic in $L + L_+$ such that

$$X_-(t) = X_+(t)F(t)u(t)F(t)^{-1}$$
$$= F'(t)u(t)F(t)^{-1}. \quad (1.14)$$

The boundary conditions are

$$X_+(0) = I, \quad F'(0) = i\epsilon, \quad (1.15a)$$
$$X_-(t) \quad \text{is analytic at} \quad t = \infty. \quad (1.15b)$$

The solution exists and is unique.[5]

A consequence of the use of special HE gauge is that the above construction of the transformation group $\mathscr{K}_L$ is independent of the contour L: henceforth we drop the subscript L on $\mathscr{K}_L$. In fact, it is sufficient to specify that $F'(t)$ or $X_+(t)$ be analytic in the whole t plane except for two branch points joined by a cut at the zeros of $S(t)$ and satisfy conditions (1.15a) and (1.11) and that $X_-(t)$ be analytic at the two branch points of $S(t)$ and at $t = \infty$. This construction does not admit gauge transformations in **K** of the form (A5) (except for $\mathscr{E} \to \mathscr{E} + i\psi_0$, ψ_0 real constant). Also, as already mentioned, a larger subgroup of **K** can be represented by relaxing the boundary conditions at $t = \infty$ (see Sec. 4).

2. THE HOMOGENEOUS HILBERT PROBLEM OF BELINSKII AND ZAKHAROV AND THE FORMULA FOR $G_0(\lambda)$ IN TERMS OF $u(t)$

The pure-nonsoliton part of the inverse scattering technique of Belinskii and Zakharov[2] is a representation of a subgroup of **K** (at least as large as $\mathscr{K}$) in the form of an homogeneous Hilbert problem qualitatively similar to that of Hauser and Ernst.[4,5] In this section, we determine the exact quantitative relationship between the BZ and HE formalisms, but first we must set up the correct boundary conditions for the BZ HHP and establish the analyticity properties of the BZ eigenfunction $\psi(\lambda)$. The spectral parameter λ is related to t by the quadratic transformation discussed in the first paragraph of Appendix B.

The respresenting matrix in the BZ formalism is a nondegenerate (i.e., possessing an inverse) 2×2 matrix function $G_0(\lambda)$, which is a function of α, β, and λ such that, when written as a function of α, β, and t using (B3a,b) *is a function of t only*,[31] and is real for real t. We shall *not* use the BZ conditions,[2]

$$G_0(\lambda) \quad \text{is symmetric}, \quad G_0(\infty) = I, \quad (2.1a,b)$$

as the former will be incompatible with our choice of gauge and the latter is unnecessarily restrictive: instead, we shall derive alternative conditions from first principles. We shall in future write $G_0(t)$ for this representing matrix, with the understanding that t is given by the right-hand side of Eq. (B1). $G_0(t)$ may have (α,β)-independent singularities anywhere in the complex t plane, but not branch points at the zeros of $S(t)$. Choose an (α,β) domain with $|\beta| > \alpha$ such that $G_0(t)$ is analytic on the circle $|\lambda| = \alpha$, to be denoted Γ.[32] Let the interior of Γ be denoted Γ_2, exterior Γ_1.

The BZ eigenfunction $\psi(\lambda)$, a function of α, β, and λ or t, is related to $F(t)$ by[16]

$$\psi(\lambda) = t^{-1}S(t)\,\mathrm{Re}\,F(t), \quad (2.2)$$

for $|\lambda| < \alpha$, and its analytic continuation to $|\lambda| \geqslant \alpha$ is determined in Appendix B. In particular,

$$\psi(0) = \mathbf{g} = \mathrm{Re}\,H. \quad (2.3)$$

We impose special HE gauge on $F(t)$. As a result, $\psi(\lambda)$ is analytic everywhere in Γ_2 and on Γ. Furthermore, condition (1.11) implies

$$\psi(\lambda)\begin{pmatrix}1 & 0\\ 0 & (\lambda-\lambda_2)^{-1}\end{pmatrix} \quad \text{is analytic at } \lambda = \lambda_2, \quad (2.4)$$

where λ_2 is the zero of $\lambda^2 + 2\beta\lambda + \alpha^2$ in Γ_2, and λ_1 is the other zero in Γ_1. If $\mathscr{E}$ is sufficiently well behaved at $(\alpha,\beta) = (0,0) = (\rho,z)$, then a similar condition holds at $\lambda = \lambda_1$. Condition (2.4) will later necessitate a boundary condition on $G_0(t)$ at $t = \infty$ (i.e., at $\lambda = \lambda_{1,2}$):

$$\begin{pmatrix}1 & 0\\ 0 & t^{-1}\end{pmatrix} G_0(t) \begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix} \quad \text{is analytic at } t = \infty. \quad (2.5)$$

The BZ HHP is to find $\chi_1(\lambda)$ analytic and nondegenerate in $\Gamma + \Gamma_1$ and at $\lambda = \infty$ and $\chi_2(\lambda)$ analytic and nondegenerate in $\Gamma + \Gamma_2$, such that, on Γ,

$$\chi_1(\lambda) = \chi_2(\lambda)G(\lambda), \quad (2.6a)$$

where

$$G(\lambda) = \psi(\lambda)G_0(t)\psi(\lambda)^{-1}. \quad (2.6b)$$

We use a prime to denote transformed variables. The new metric $\mathbf{g}'$ and eigenfunction $\psi'(\lambda)$ in special HE gauge are given by

$$\psi'(\lambda) = \chi_2(\lambda)\psi(\lambda), \quad \mathbf{g}' = \psi'(0) = \chi_2(0)\mathbf{g}. \quad (2.7a,b)$$

In addition to the HHP, Belinskii and Zakharov also impose an auxiliary functional relation,

$$\mathbf{g}' = \chi_1(\alpha^2/\lambda)\mathbf{g}\chi_2(\lambda)^T, \quad (2.8)$$

^{T} denoting matrix transpose, which is compatible with the HHP and our choice of gauge. [It will be seen to be related to Eq. (75) of Ref. 4.] The transpose of Eq. (2.8) is

$$\mathbf{g}' = \chi_2(\lambda)\mathbf{g}\chi_1(\alpha^2/\lambda)^T. \quad (2.9)$$

Now put $\lambda = 0$ in Eq. (2.9) and compare with Eq. (2.7b): we find

$$\chi_1(\infty) = I. \quad (2.10)$$

This the boundary condition at $\lambda = \infty$ that we shall use. It is intended to replace the BZ conditions (2.1b) and $\chi_2(\infty) = I$, which are more restrictive. In fact, we do not require $G_0(t)$ or $\chi_2(\lambda)$ to be analytic at $\lambda = \infty$, and so the HHP is not necessarily "canonical."[26]

The *mixed soliton–nonsoliton* transformations[2] are solutions of a generalization of the above HHP in which $\chi_2(\lambda)$ is allowed to have poles at specified points, $\lambda = \mu(s_1),\ldots,\mu(s_n)$, in Γ_2 and so, from Eq. (2.8), $\chi_1(\lambda)$ is degenerate at $\lambda = \nu(s_1),\ldots,\nu(s_n)$ in Γ_1 [see Eqs. (B3a,b)]. The solution of this generalized HHP is not unique and it is necessary to resort to the differential equation[2] for $\psi'(\lambda)$ to identify the residues of $\chi_2(\lambda)$ at the poles in Γ_2. The *pure-soliton* transformations, as yet the only case discussed in any detail in the literature,[2,15,16] are the special cases of the mixed transformations for which

$$G_0(t) = I, \quad \chi_1(\lambda) = \chi_2(\lambda). \quad (2.11)$$

Before discussing the determinant of $G_0(t)$, we must recall from Ref. 2 that the BZ formalism does not guarantee the relations

$$\det \mathbf{g}' = \alpha^2, \quad \det \boldsymbol{\psi}'(\lambda) = \lambda^2 + 2\beta\lambda + \alpha^2 = \lambda/t \tag{2.12a,b}$$

for the transformed solution. In fact, it is well known that Eqs. (2.12a,b) do not hold in the pure-soliton case and so could not hold, in general, in the mixed soliton–nonsoliton case. In any case, however, a "physical" metric and eigenfunction satisfying *all* of the field equations and eigenvalue equations can be constructed from the rescalings[2,16]:

$$\mathbf{g}'_{\text{ph}} = \alpha(\det \mathbf{g}')^{-1/2}\mathbf{g}', \tag{2.13a}$$

$$\boldsymbol{\psi}'_{\text{ph}} = (\lambda/t)^{1/2}(\det \boldsymbol{\psi}')^{-1/2}\boldsymbol{\psi}' = (\det \boldsymbol{\chi}_2)^{-1/2}\boldsymbol{\chi}_2\boldsymbol{\psi}. \tag{2.13b}$$

It is easy to see that Eqs. (2.12a,b) hold in the pure-nonsoliton case provided

$$\det G_0(t) = 1. \tag{2.14}$$

We shall show that condition (2.14) can be imposed on $G_0(t)$ without loss of generality in all cases. The nonmatrix HHP,

$$\theta_1(\lambda) = \theta_2(\lambda)\det G_0(t), \quad \theta_1(\infty) = 1, \tag{2.15}$$

$\theta_{1,2}$ analytic and nonvanishing in $\Gamma_{1,2}$, respectively, has the unique solution,

$$\theta_{1,2}(\lambda) = \exp\left\{ -\frac{1}{2\pi i}\int_\Gamma \frac{D(\lambda')}{\lambda' - \lambda}\, d\lambda' \right\}, \quad \lambda\in\Gamma_{1,2}, \tag{2.16a}$$

where

$$D(\lambda) = \ln\det G_0(t) = D(\alpha^2/\lambda) \tag{2.16b}$$

[note that no branch cuts of $D(\lambda)$ intersect the circle Γ]. The change of variable, $\lambda' \to \alpha^2/\lambda''$, in Eqs. (2.16a) yields the functional relation,

$$\theta_1(\alpha^2/\lambda) = \theta_2(0)/\theta_2(\lambda). \tag{2.17}$$

Define

$$\overline{G}_0(t) = [\det G_0(t)]^{-1/2}G_0(t), \quad \overline{G}(\lambda) = \boldsymbol{\psi}(\lambda)\overline{G}_0(t)\boldsymbol{\psi}(\lambda)^{-1}, \tag{2.18a,b}$$

$$\bar{\boldsymbol{\chi}}_1(\lambda) = \theta_1{}^{-1/2}(\lambda)\boldsymbol{\chi}_1(\lambda), \quad \bar{\boldsymbol{\chi}}_2(\lambda) = \theta_2{}^{-1/2}(\lambda)\boldsymbol{\chi}_2(\lambda), \tag{2.18c,d}$$

and observe that $\det \overline{G}_0(t) = 1 = \det \overline{G}(\lambda)$. Then the HHP for the pure-nonsoliton or mixed cases can be rewritten in terms of barred quantities:

$$\bar{\boldsymbol{\chi}}_1(\lambda) = \bar{\boldsymbol{\chi}}_2(\lambda)\overline{G}(\lambda), \tag{2.19a}$$

$$\bar{\boldsymbol{\psi}}'(\lambda) = \bar{\boldsymbol{\chi}}_2(\lambda)\boldsymbol{\psi}(\lambda) = \theta_2{}^{-1/2}(\lambda)\boldsymbol{\psi}'(\lambda), \tag{2.19b}$$

$$\bar{\mathbf{g}}' = \bar{\boldsymbol{\chi}}_1(\alpha^2/\lambda)\mathbf{g}\bar{\boldsymbol{\chi}}_2(\lambda)^T = \theta_2{}^{-1/2}(0)\mathbf{g}'. \tag{2.19c}$$

In the pure-nonsoliton case, we must identify

$$\theta_{1,2}(\lambda) = \det \boldsymbol{\chi}_{1,2}(\lambda), \tag{2.20}$$

and so

$$\det \bar{\boldsymbol{\chi}}_{1,2}(\lambda) = 1, \quad \bar{\mathbf{g}}' = \mathbf{g}'_{\text{ph}}, \quad \bar{\boldsymbol{\psi}}'(\lambda) = \boldsymbol{\psi}'_{\text{ph}}(\lambda). \tag{2.21a,b,c}$$

In the mixed or pure-soliton cases, we must still use Eqs. (2.13a,b) with barred quantities on the right-hand sides. In these cases, Eqs. (2.19a,c) imply

$$\Delta \equiv \det \bar{\boldsymbol{\chi}}_{1,2}(\lambda) = \frac{[\lambda - \nu(s_1)][\lambda - \nu(s_2)]\ldots[\lambda - \nu(s_n)]}{[\lambda - \mu(s_1)][\lambda - \mu(s_2)]\ldots[\lambda - \mu(s_n)]}, \tag{2.22}$$

in agreement with Eq. (5.16) of Ref. 16. In future, we accept condition (2.14) and drop the bars.

We now turn to the determination of the correct symmetry condition on the representing matrix $G_0(t)$. This will be deduced from the HHP (2.6a,b), the auxiliary relation (2.8), and the following functional relations which are proved in Appendices A and B:

$$\widetilde{F}(t) = iF(t)h(t), \tag{2.23}$$

$$\boldsymbol{\psi}(\alpha^2/\lambda) = \lambda^{-1}\mathbf{g}\boldsymbol{\epsilon}\boldsymbol{\psi}(\lambda)h(t), \tag{2.24}$$

where

$$h(t) = \begin{pmatrix} -\psi f^{-1} & -t^{-1}f^{-1} \\ t(f+\psi^2 f^{-1}) & \psi f^{-1} \end{pmatrix} \tag{2.25}$$

and $f = \operatorname{Re}\mathscr{E}$, $\psi = \operatorname{Im}\mathscr{E}$, evaluated at $\alpha = 0, \beta = (2t)^{-1}$. The matrix $h(t)$ obeys the relations (A12). In Eq. (2.23), $\widetilde{F}(t)$ is the value of the analytic continuation of $F(t)$ on the second Riemann sheet. [The validity of these relations depends on analyticity of $\mathscr{E}(\alpha,\beta)$ at the origin (0,0), which can be brought about by a time translation, $\beta\to\beta$ + constant (or axial translation, $z\to z$ + constant, in the stationary case), if $\mathscr{E}$ is analytic at at least one point of the axis.]

Now, use Eq. (2.6a) to eliminate $\boldsymbol{\chi}_1$ in Eqs. (2.8) and (2.9) and replace λ by α^2/λ in the latter: the results are

$$\mathbf{g}' = \boldsymbol{\chi}_2(\alpha^2/\lambda)G(\alpha^2/\lambda)\mathbf{g}\boldsymbol{\chi}_2(\lambda)^T, \tag{2.26a}$$

$$\mathbf{g}' = \boldsymbol{\chi}_2(\alpha^2/\lambda)\mathbf{g}G(\lambda)^T\boldsymbol{\chi}_2(\lambda)^T. \tag{2.26b}$$

Comparison of these two equations gives

$$G(\alpha^2/\lambda)\mathbf{g} = \mathbf{g}G(\lambda)^T. \tag{2.27}$$

Next, use Eqs. (2.6b), (2.14), (2.24), and (B14) and observe that $\mathbf{g}$ and $\boldsymbol{\psi}(\lambda)$ can be cancelled from the final equation [remember that $G_0(t)$ is unaffected by the replacement, $\lambda\to\alpha^2/\lambda$]. The result is

$$h(t)G_0(t)h(t)^{-1} = G_0(t)^{-1}. \tag{2.28}$$

From Eqs. (A12), this can be written

$$[G_0(t)h(t)]^2 = -I, \tag{2.29}$$

which itself implies

$$\operatorname{tr}[G_0(t)h(t)] = \operatorname{tr}[h(t)G_0(t)] = 0. \tag{2.30}$$

Since the product of a traceless matrix with $\boldsymbol{\epsilon}$ in either order is symmetric and vice versa, it follows that Eq. (2.30) expresses the required symmetry condition on $G_0(t)$.

We are now in a position to establish the exact relationship between the BZ nonsoliton and HE formalisms. Start again with Eq. (2.14a) and use Eqs. (2.7a), (2.6b), (2.24), and (B14) to obtain an equation in which $\mathbf{g}$, $\boldsymbol{\psi}(\lambda)$ and their transforms can be cancelled out. The result is

$$h'(t)G_0(t)h(t)^{-1} = I, \tag{2.31}$$

where $h'(t)$ is calculated from Eq. (2.25) with primed variables. This equation, which can be written

$$h'(t) = h(t)G_0(t)^{-1} = G_0(t)h(t), \tag{2.32}$$

is in fact a formula for the transformed Ernst potential $\mathscr{E}'$ on the axis $\alpha = 0 = \rho$, according to Eq. (2.25). It is an important formula in its own right and, incidentally, shows that $G_0(t)$ is a genuine representation of $\mathscr{K}$ in that the group product is represented by the matrix product. An alternative interpretation of Eq. (2.32) is that it provides a formula for $G_0(t)$ in terms of the initial and final values of the Ernst potential on the axis, thereby demonstrating the transitivity of $\mathscr{K}$ on the space of solutions analytic at and near $(\alpha,\beta) = (0,0)$. Already, Hauser and Ernst have derived just such a formula for their representing matrix $u(t)$[5]:

$$(t\mathscr{E}', \ i)u(t)\begin{pmatrix} -it^{-1} \\ \mathscr{E} \end{pmatrix} = 0, \tag{2.33}$$

where $\mathscr{E}$ and $\mathscr{E}'$ are to be evaluated at $\alpha = 0 = \rho$, $\beta = (2t)^{-1} = z$. Indeed, Eq. (2.33) has already been used in the derivation of Eq. (2.25) in Appendix A.

By expressing $\mathscr{E}'$ as the subject in Eq. (2.33) and separating into real and imaginary parts, we get from Eq. (2.25) the equivalent formula,

$$h'(t) = u(t)h(t)u(t)^{-1}, \tag{2.34}$$

in agreement with Eq. (A15) which was derived directly from the definition (A11) of $h(t)$. Finally, comparison of Eqs. (2.32) and (2.34) reveals the desired relationship between $G_0(t)$ and $u(t)$:

$$G_0(t) = u(t)h(t)u(t)^{-1}h(t)^{-1}. \tag{2.35}$$

Clearly, conditions (2.5), (2.14), and (2.30) [or (2.29)] on $G_0(t)$ are satisfied identically.

Now, if one wishes to calculate the transform of $\mathscr{E}$ and $F(t)$ under an element of $\mathscr{K} \subset \mathbf{K}$ represented by $u(t)$, one can, in principle, substitute Eq. (2.35) into Eqs. (2.6a,b) and solve the BZ homogeneous Hilbert problem. However, a serious difficulty arises with a large class of $u(t)$ matrices for which the HE HHP can be solved explicitly in closed form. These $u(t)$ matrices have only poles and/or quadratic branch points[11] and represent finite products of null generalized[16] HKX transformations[12] and Harrison transformations.[13,14] The HE HHP can be solved without a detailed knowledge of the analytic behavior of $F(\rho,z,t)$ or $\mathscr{E}(\rho,z)$ and the resulting expression for $F'(t)$ is a rational function of $F(t)$ and the values of $F(t)$ and possibly some of its t-derivatives at the singular points of $u(t)$. The same methods will not work for the BZ HHP in its present form because of the appearance of $h(t)$ which depends on the analytic behavior of $\mathscr{E}(0,(2t)^{-1})$ in the complex t plane. In the next paragraph, we provide an alternative formulation of the BZ HHP in which $h(t)$ does not appear.

Substitute Eq. (2.35) into Eqs. (2.6a,b) to give

$$\chi_1(\lambda) = \chi_2(\lambda)\psi(\lambda)u(t)h(t)u(t)^{-1}h(t)^{-1}\psi(\lambda)^{-1}. \tag{2.36}$$

Define

$$Y(\lambda) = \psi'(\lambda)u(t)\psi(\lambda)^{-1}$$
$$= \chi_2(\lambda)\psi(\lambda)u(t)\psi(\lambda)^{-1}. \tag{2.37a}$$

Then, from Eqs. (2.36) and (2.24),

$$Y(\lambda) = \chi_1(\lambda)\mathbf{g}\boldsymbol{\epsilon}\psi(\alpha^2/\lambda)u(t)\psi(\alpha^2/\lambda)^{-1}(\mathbf{g}\boldsymbol{\epsilon})^{-1}. \tag{2.37b}$$

The new form of the HHP is expressed by equating the right-hand sides of Eqs. (2.37a) and (2.37b). Also, in terms of the matrix function $Y(\lambda)$, the auxiliary relation (2.8) reads

$$\mathbf{g}' = Y(\alpha^2/\lambda)\mathbf{g}Y(\lambda)^T. \tag{2.38}$$

Observe that all the factors in Eq. (2.37a) are analytic in $\Gamma_2 - \{\lambda_2\}$ except $u(t)$ and all the factors in Eq. (2.37b) are analytic in $\Gamma_1 - \{\lambda_1\}$ except $u(t)$. [Conditions (1.13d) and (2.4) guarantee that $Y(\lambda)$ is analytic at both $\lambda = \lambda_1, \lambda_2$.] This fact allows the new HHP to be solved in closed form for the $u(t)$ which have only poles and quadratic branch points as in Ref. 11. [Of course, the HHP (2.6a,b) can be solved by the same methods in a large number of cases where both $F(t)$ and $G_0(t)$ are specified.] In Sec. 3, we derive the Harrison and HKX transformations from the HHP (2.37a,b) by methods comparable to those of Ref. 11, and we also generate the general Weyl (Einstein–Rosen) solution from flat space using the original form (2.6a,b) of the BZ HHP.

The unknowns $X_+(t)$ and $X_-(t)$ in the HE HHP can be expressed in terms of the BZ unknowns $\chi_2(\lambda)$ and $\chi_1(\lambda)$ by the following symmetric relations:

$$X_+(t) = \frac{\lambda}{\alpha^2 - \lambda^2}[-\chi_2(\lambda)(\lambda I - i\mathbf{g}\boldsymbol{\epsilon}) + \chi_1(\alpha^2/\lambda)(\alpha^2\lambda^{-1}I - i\mathbf{g}\boldsymbol{\epsilon})], \tag{2.39a}$$

$$X_-(t) = \frac{\lambda}{\alpha^2 - \lambda^2}[-Y(\lambda)(\lambda I - i\mathbf{g}\boldsymbol{\epsilon}) + Y(\alpha^2/\lambda)(\alpha^2\lambda^{-1}I - i\mathbf{g}\boldsymbol{\epsilon})], \tag{2.39b}$$

$\lambda \in \Gamma_2$ in (2.39a), no restriction on λ in (2.39b). These equations show directly that $X_+(t)$ is analytic in the region Γ_2 of the λ plane, and $X_-(t)$ is invariant under the replacement $\lambda \to \alpha^2/\lambda$ and hence is analytic at the branch points of $S(t)$ in the t plane. The inverse relations are

$$\chi_2(\lambda) = \mathrm{Re}\, X_+(t) - \lambda^{-1}\, \mathrm{Im}\, X_+(t)\mathbf{g}\boldsymbol{\epsilon}, \quad \lambda \in \Gamma_2, \tag{2.40a}$$

$$\chi_1(\alpha^2/\lambda) = \mathrm{Re}\, X_+(t) - (\lambda/\alpha^2)\, \mathrm{Im}\, X_+(t)\mathbf{g}\boldsymbol{\epsilon}, \quad \lambda \in \Gamma_2, \tag{2.40b}$$

$$Y(\lambda) = \mathrm{Re}\, X_-(t) - \lambda^{-1}\, \mathrm{Im}\, X_-(t)\mathbf{g}\boldsymbol{\epsilon}, \quad \text{for all } \lambda. \tag{2.40c}$$

These five relations, together with Eq. (2.35), allow the BZ HHP to be deduced from the HE HHP and vice versa. In particular, the auxiliary relations (2.8) and (2.38) are immediate consequences of Eq. (75) of Ref. 4 and vice-versa. Thus the restriction on the gauge of $\psi'(\lambda)$ implied by (2.8) is compatible with the original gauge restrictions on the potentials of Ref. 8. If mixed soliton–nonsoliton transformations are under consideration, $\chi_{1,2}(\lambda)$ and $Y(\lambda)$ in the above formulas should be replaced by $\Delta^{-1/2}\chi_{1,2}(\lambda)$ and $\Delta^{-1/2}Y(\lambda)$, respectively, where Δ is given by Eq. (2.22).

There is an important qualitative difference between the group representations denoted by $u(t)$ and $G_0(t)$. First, $u(t)$ can serve as a representing matrix for an abstract Lie group which exists independently of solutions of Einstein's equations, whereas $G_0(t)$ cannot: condition (2.30) explicitly involves the Ernst potential $\mathscr{E}$ through $h(t)$. Further, since $G_0(t)$ is uniquely determined in terms of the initial and final solutions by Eq. (2.32), it follows that $G_0(t)$ does not faithfully represent the multiple transitivity of the $\mathscr{K}$-group. We can say that $G_0(t)$ faithfully represents a simply transitive factor group in which each element is an equivalence class of members of $\mathscr{K}$ which transform any initial solution to the same final solution. The equivalence class for each $G_0(t)$ can be expressed by solving Eq. (2.35) for $u(t)$: the result is

$$u(t) = G_0(t)^{1/2}\{[\cos\theta(t)]I + [\sin\theta(t)]h(t)\}, \qquad (2.41)$$

where $\theta(t)$ is an arbitrary function of t, real for real t, and analytic in $L + L_-$ and at $t = \infty$, and where

$$G_0(t)^{1/2} = D^{-1/2}[I + G_0(t)], \qquad (2.42a)$$

$$D = \det[I + G_0(t)] = 2 + \operatorname{tr} G_0(t). \qquad (2.42b)$$

The transformations in $\mathscr{K}$ which preserve asymptotic flatness in the stationary axisymmetric case can be identified from Eq. (2.33) which expresses $\mathscr{E}'(0,z)$ in terms of $\mathscr{E}(0,z)$ and $u(t)$. A simple calculation shows that a *necessary and sufficient condition for preservation of asymptotic flatness* up to a NUT parameter is

$$\begin{pmatrix}1 & 0\\ 0 & t^{-1}\end{pmatrix} u(t) \begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix} \text{ is analytic at } t = 0. \qquad (2.43a)$$

According to Eqs. (2.35) and (2.25), an equivalent statement is

$$\begin{pmatrix}1 & 0\\ 0 & t^{-1}\end{pmatrix} G_0(t) \begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix} \text{ is analytic at } t = 0. \qquad (2.43b)$$

The definition of asymptotic flatness assumed here is given in Appendix B of Ref. 16. It must be emphasized how trivially easy it is to identify asymptotic flatness preserving transformations in terms of representing matrices. This powerful feature allows the investigator to work confidently with more general transformations in $\mathscr{K}$ since it is easy to recognize combinations of the latter which preserve asymptotic flatness (e.g., even number of Harrison transformations) and to identify members of $\mathscr{K}$, if any, which map a given asymptotically nonflat solution to an asymptotically flat solution (e.g., single Harrison transform of the "stationary C metric").

3. EXAMPLES

In this section, we investigate the BZ homogeneous Hilbert problem (2.6a, b) or its alternative form (2.37a, b) in four special cases: (A) Harrison's Bäcklund transformation,[13,14,16] (B) the null generalized HKX transformation,[12,16] (C) generation of the general Weyl (Einstein–Rosen) solution from flat space,[9] and (D) generation of the nonlinear superposition of n Kerr–NUT particles from the superposition of n Schwarzschild particles using the **B** group.[10] The first three examples have already been treated in the HE representation (Refs. 11, 3 and 11, and 4, respectively) and it is instructive to compare the calculations. Also, Hauser and Ernst have generated Kerr–NUT from Schwarzschild using their integral-equation representation of the **B** group.[3] In Sec. 3D, we discuss the **B** group from the standpoint of the BZ and HE HHP's and we find that the BZ representation of the n-Schwarzschild→n-Kerr–NUT transformation is different from the others, being of *pure-soliton* type.

A. Harrison's Bäcklund transformation

Let us solve the BZ HHP written in the alternative form of Eqs. (2.37a, b),

$$\begin{aligned}&\boldsymbol{\psi}'(\lambda)u(t)\boldsymbol{\psi}(\lambda)^{-1}\\ &\quad= \boldsymbol{\chi}_1(\lambda)\mathbf{g}\boldsymbol{\epsilon}\boldsymbol{\psi}(\alpha^2/\lambda)u(t)\boldsymbol{\psi}(\alpha^2/\lambda)^{-1}(\mathbf{g}\boldsymbol{\epsilon})^{-1},\end{aligned} \qquad (3.1)$$

for[11]

$$u(t) = \left(1 - \frac{s}{t}\right)^{-1/2}\begin{pmatrix}1 & -cst^{-1}\\ -c^{-1} & 1\end{pmatrix}. \qquad (3.2)$$

The unknowns are $\boldsymbol{\psi}'(\lambda)$ analytic in Γ_2 and obeying condition (2.4) and $\boldsymbol{\chi}_1(\lambda)$ analytic in Γ_1 and obeying condition (2.10). Write

$$\begin{aligned}(\lambda - \mu_2)(\lambda - \mu_1) &= \lambda^2 + (2\beta - s^{-1})\lambda + \alpha^2\\ &= \lambda(st)^{-1}(s - t),\end{aligned}$$

with $\mu_2 = \mu(s) \in \Gamma_2$, $\mu_1 = \nu(s) \in \Gamma_1$. Since $u(t)$ appears linearly and homogeneously in Eq. (3.1), the quadratic surd $(1 - s/t)^{-1/2}$ cancels out and so $u(t)$ can be replaced by the rational matrix function,

$$v(t) = \begin{pmatrix}1 & -c[1 + s\lambda^{-1}(\lambda - \mu_2)(\lambda - \mu_1)]\\ -c^{-1} & 1\end{pmatrix}, \qquad (3.3)$$

which has poles at $\lambda = 0, \infty$ and a vanishing determinant at $\lambda = \mu_2, \mu_1$.

Now, $\boldsymbol{\psi}'(\lambda)v(t)\boldsymbol{\psi}(\lambda)^{-1}$ is analytic throughout Γ_2 (including $\lambda = \lambda_2$ and $\lambda = \mu_2$) except for the origin $\lambda = 0$ where it has a simple pole with residue,

$$\mathbf{g}'\begin{pmatrix}0 & -cs\alpha^2\\ 0 & 0\end{pmatrix}\mathbf{g}^{-1} = cs\begin{pmatrix}f'_{11}\\ f'_{12}\end{pmatrix}(f_{12}, \ -f_{11}). \qquad (3.4)$$

Similarly, $\boldsymbol{\chi}_1(\lambda)\mathbf{g}\boldsymbol{\epsilon}\boldsymbol{\psi}(\alpha^2/\lambda)v(t)\boldsymbol{\psi}(\alpha^2/\lambda)^{-1}(\mathbf{g}\boldsymbol{\epsilon})^{-1}$ is analytic throughout Γ_1 (including $\lambda = \lambda_1$ and $\lambda = \mu_1$) and grows linearly as $\lambda \to \infty$, say $\sim \mathbf{e}\lambda$, where

$$\mathbf{e} = cs\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

It follows that

$$\begin{aligned}\boldsymbol{\psi}'(\lambda)v(t)\boldsymbol{\psi}(\lambda)^{-1} &= \boldsymbol{\chi}_1(\lambda)\mathbf{g}\boldsymbol{\epsilon}\boldsymbol{\psi}(\alpha^2/\lambda)v(t)\boldsymbol{\psi}(\alpha^2/\lambda)^{-1}(\mathbf{g}\boldsymbol{\epsilon})^{-1}\\ &= A\lambda^{-1} + B + cs\lambda\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \equiv Z(\lambda),\end{aligned} \qquad (3.5)$$

where A and B are matrices depending only on (α, β), to be determined.

From Eq. (3.5),

$$\boldsymbol{\psi}'(\lambda) = \frac{\lambda}{s(\lambda - \mu_2)(\lambda - \mu_1)} Z(\lambda)\boldsymbol{\psi}(\lambda)\boldsymbol{\epsilon}v(t)^T\boldsymbol{\epsilon}, \qquad (3.6)$$

$$\begin{aligned}\boldsymbol{\chi}_1(\lambda) = -&\frac{\lambda}{s(\lambda - \mu_2)(\lambda - \mu_1)} Z(\lambda)\mathbf{g}(\boldsymbol{\psi}(\alpha^2/\lambda)^{-1})^T\\ &\times v(t)^T\boldsymbol{\psi}(\alpha^2/\lambda)^T\mathbf{g}^{-1},\end{aligned} \qquad (3.7)$$

where we have used the identity (B14). The right-hand side of Eq. (3.6) must not have a pole at $\lambda = \mu_2$ in Γ_2 and the right-hand side of Eq. (3.7) must not have a pole at $\lambda = \mu_1$ in Γ_1. Since $v(t)$ is degenerate at $\lambda = \mu_1$ and $\lambda = \mu_2$, the conditions that the respective residues vanish reduce to the following column-vector equations:

$$Z(\mu_2)\boldsymbol{\psi}(\mu_2)\begin{pmatrix}c\\ 1\end{pmatrix} = 0, \qquad (3.8)$$

$$Z(\mu_1)\mathbf{g}\boldsymbol{\epsilon}\boldsymbol{\psi}(\mu_2)\begin{pmatrix}c\\ 1\end{pmatrix} = 0. \qquad (3.9)$$

These equations provide only four equations for the eight unknown entries in A and B. Since A is given by either side of Eq. (3.4), we have

$$A\begin{pmatrix} f_{11} \\ f_{12} \end{pmatrix} = fA\begin{pmatrix} 1 \\ -\omega \end{pmatrix} = 0 \tag{3.10}$$

[recall Eq. (1.4)]. A fourth column-vector equation can be deduced from the requirement $\chi_1(\infty) = I$. Taking the limit as $\lambda \to \infty$ of the right-hand side of Eq. (3.7), we find

$$B\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c^{-1} \\ 1 \end{pmatrix} + cs\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{g}\boldsymbol{\epsilon}\dot{\boldsymbol{\psi}}(0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c^{-1} \\ 1 + cs\psi \end{pmatrix}, \tag{3.11}$$

where $\psi = \operatorname{Im} \mathscr{E}$. In deriving Eq. (3.11), we have used Eqs. (B8b), (B5), (A6b), and (1.8).

The solution of Eqs. (3.8)–(3.11) for the matrices A and B is now a straightforward problem in linear algebra. The result, after some rearrangement, is

$$A = \begin{pmatrix} c^{-1}\operatorname{Im} T & 0 \\ \operatorname{Im}[(1 - ics\mathscr{E})T] & csf \end{pmatrix} \mathbf{g}\boldsymbol{\epsilon} + cs\alpha^2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3.12a}$$

$$B = \begin{pmatrix} -c^{-1}\operatorname{Re} T & c^{-1} \\ -\operatorname{Re}[(1 - ics\mathscr{E})T] & 1 + cs\psi \end{pmatrix} - c(1 - 2s\beta)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3.12b}$$

where the complex pseudopotential T is defined by

$$T = T_{(\mathrm{N})}/T_{(\mathrm{D})}, \tag{3.13a}$$

$$\begin{pmatrix} T_{(\mathrm{D})} \\ T_{(\mathrm{N})} \end{pmatrix} = (\mu_1 - \mu_2)^{-1}(I + i\mu_2{}^{-1}\mathbf{g}\boldsymbol{\epsilon})\boldsymbol{\psi}(\mu_2)\begin{pmatrix} c \\ 1 \end{pmatrix}. \tag{3.13b}$$

Hence, from Eqs. (3.6), (3.5), and (3.3), the solution of the HHP is

$$\boldsymbol{\psi}'(\lambda) = -[s(\lambda - \mu_2)(\lambda - \mu_1)]^{-1}\left\{A + \lambda B + cs\lambda^2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\} \times \boldsymbol{\psi}(\lambda)\begin{pmatrix} 1 & c[1 + s\lambda^{-1}(\lambda - \mu_2)(\lambda - \mu_1)] \\ c^{-1} & 1 \end{pmatrix}. \tag{3.14}$$

The limiting form of this equation as $\lambda \to 0$ gives the transformed metric

$$\mathbf{g}' = \boldsymbol{\psi}'(0) = \begin{pmatrix} f'_{11} & f'_{12} \\ f'_{12} & f'_{22} \end{pmatrix}, \tag{3.15}$$

where

$$f'_{11} = -(1/c^2 s)\operatorname{Im} T, \tag{3.16a}$$

$$f'_{12} = -(1/cs)\operatorname{Im}[(1 - ics\mathscr{E})T] + f\omega, \tag{3.16b}$$

$$f'_{22} = -(1/s)\operatorname{Im}[(1 - ics\mathscr{E})^2 T] + 2cf\omega(1 + cs\psi) + c^2 f(1 - 2s\beta). \tag{3.16c}$$

It can now be shown directly that Eqs. (2.8) [or (2.38)] and (2.12a, b) are satisfied identically.

It is not difficult to obtain the transformed complex $F(t)$ potential from $\boldsymbol{\psi}'(\lambda)$ and $\mathbf{g}'$. The relevant formulas are in Appendix B. Restrict λ to the open disk Γ_2 so that $\lambda = \mu(t)$, $\mu_2 = \mu(s)$, $\mu_1 = \nu(s)$. We find

$$T_{(\mathrm{N})} = F_{22}(s) + cF_{21}(s), \quad T_{(\mathrm{D})} = F_{12}(s) + cF_{11}(s), \tag{3.17}$$

$$F'(t) = \frac{t}{t - s}\begin{pmatrix} -c^{-1}T & c^{-1} \\ c(s - t)t^{-1} - (1 - ics\mathscr{E})T & 1 - ics\mathscr{E} \end{pmatrix} \times F(t)\begin{pmatrix} 1 & cst^{-1} \\ c^{-1} & 1 \end{pmatrix}, \tag{3.18}$$

in exact agreement with Refs. 11 and 16.

The principal difference between the derivation of Eq. (3.18) here for the Harrison transformation and the derivation of the same equation in Ref. 11 [Eq. (3.21)] arises from the fact that each singularity of $u(t)$ or $v(t)$, or degeneracy of $v(t)$, in the t plane corresponds to two singularities in the λ plane, one in Γ_2, the other in Γ_1. Thus the number of unknown coefficients to be determined in the BZ HHP is exactly double the number in the HE HHP. The same comment applies to the product of several Harrison transformations, including confluent cases such as the null HKX transformation which is the product of two Harrison transformations with same s parameters.[11,16] Thus the HE HHP leads to shorter computations in these cases. An additional advantage of the HE formalism is the use of complex potentials which also often lead to shorter expressions.

B. The null HKX transformation

In Ref. 11, we solved the HE HHP for the $u(t)$ given by the following SL(2)-covariant expression,

$$u(t) = u^A{}_B(t) = -\epsilon^A{}_B + [at/(t - s)]q^A q_B, \tag{3.19}$$

where a, s and q^A ($A = 1, 2$) are real constants, a being a canonical group parameter for fixed s and q^A. [In SL(2)-tensor equations, $\epsilon^{AB} = \boldsymbol{\epsilon} = \epsilon_{AB}$ is the index raising and lowering operator, e.g., $q_B = q^X\epsilon_{XB}$, $q^A = \epsilon^{AX}q_X$, and, in particular, $\epsilon_B{}^A = -\epsilon^A{}_B = \delta^A_B$.] The result was the null generalized HKX transformation,[11,16,33]

$$\mathrm{F}'_{AB}(t) = \left\{F_A{}^X(t) + \frac{aq^C q^D F_{AC}(s)[G_D{}^X(s, t) - s(s - t)^{-1}\epsilon_D{}^X]}{1 - aq^E q^F G_{EF}(s, s)}\right\} g_{XB}(t), \tag{3.20}$$

where

$$G_{AB}(s, t) = \frac{s}{s - t}\epsilon_{AB} + \frac{tS(s)}{s - t}F_{XA}(s)F^X{}_B(t), \tag{3.21}$$

$$g_{AB}(t) = u_{BA}(t) \quad \text{or} \quad g(t) = u(t)^{-1} \tag{3.22}$$

[recall $g(t) = -g^A{}_B(t)$, $g_{AB}(t) = \boldsymbol{\epsilon}g(t)$, and Eq. (B14)]. This transformation preserves modified special HE gauge[30] since $u(t)$ is analytic at and near $t = \infty$ but does not satisfy condition (1.13d), except when $q^1 = 0$. The factor $g_{XB}(t)$ on the right-hand side of Eq. (3.20) is a gauge function since $g(t)[= -g^A{}_B(t)]$ satisfies conditions (A8). If this gauge function is deleted [i.e., $g_{XB}(t) = \epsilon_{XB}$ or $g(t) = I$], the resulting transformation is denoted the "extended" HKX transformation.[16] This transformation is of pure-soliton type and the close relationship between certain pure-soliton and pure-nonsoliton transformations is discussed at the end of this subsection.

In order to derive the null HKX transformation (3.20) from the BZ nonsoliton formalism, we must abandon SL(2) covariance and insist on special HE gauge and condition (1.13d). These requirements can be met by multiplying the $u(t)$ of Eq. (3.19) on the left by[30]

$$\mathbf{b} = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}, \quad b = \frac{a(q^1)^2}{1 + aq^1q^2}. \tag{3.23}$$

The resulting representing matrix is

$$\bar{u}(t) = \mathbf{b}u(t) = \begin{pmatrix} 1 - \dfrac{aq^1q^2t}{(1+aq^1q^2)(t-s)} & \dfrac{a(q^1)^2 s}{(1+aq^1q^2)(t-s)} \\ -\dfrac{a(q^2)^2t}{t-s} & 1 + \dfrac{aq^1q^2t}{t-s} \end{pmatrix}. \tag{3.24}$$

This $\bar{u}(t)$ has a simple pole at $t = s$ and hence, when written as a function of λ, it has simple poles at $\lambda = \mu_2 = \mu(s)$ in Γ_2 and $\lambda = \mu_1 = \nu(s)$ in Γ_1 as in the preceding subsection.

We take the BZ HHP in the form (2.37a, b) and observe immediately that $Y(\lambda)$ must be analytic everywhere except for simple poles at $\lambda = \mu_2, \mu_1$ only. Thus

$$Y(\lambda) = Y(\infty)\left[I + \frac{C}{\lambda - \mu_2} + \frac{D}{\lambda - \mu_1}\right], \tag{3.25}$$

where C and D are matrix functions of (α, β) to be determined, and, from the boundary condition (2.10),

$$Y(\infty) = -\boldsymbol{\epsilon}\bar{u}(0)\boldsymbol{\epsilon} = (\mathbf{b}^{-1})^T. \tag{3.26}$$

The HHP now reduces to the problem of writing down and solving eight linear equations for the eight unknown entries in C and D.

We shall not show the details of the calculation, but only how the required number of equations can be written down, for thereafter the completion of the calculation is a straightforward problem in linear algebra. [Compare with the calculation of $R_A{}^B$ in Eq. (3.4) of Ref. 11.] First, equate the residues at $\lambda = \mu_2$ on both sides of Eq. (2.37a) and the residues at $\lambda = \mu_1$ on both sides of Eq. (2.37b). We obtain expressions for the matrices C and D in terms of the column vectors

$$\boldsymbol{\chi}_2(\mu_2)\boldsymbol{\psi}(\mu_2)\begin{pmatrix} q^1 \\ q^2(1+aq^1q^2) \end{pmatrix}, \quad \boldsymbol{\chi}_1(\mu_1)\mathbf{g}\boldsymbol{\epsilon}\boldsymbol{\psi}(\mu_2)\begin{pmatrix} q^1 \\ q^2(1+aq^1q^2) \end{pmatrix},$$

respectively. To obtain linear algebraic equations for these column vectors, express $\boldsymbol{\chi}_2(\lambda)$ and $\boldsymbol{\chi}_1(\lambda)$ as the subjects in Eqs. (2.37a, b), respectively, and take the appropriate limits as $\lambda \to \mu_2, \mu_1$, respectively. The final results are

$$C = \frac{a\mu_2}{s(\mu_1 - \mu_2)}\frac{1}{m^2 + n^2}\,[mI + \mu_2{}^{-1}n\mathbf{g}\boldsymbol{\epsilon}]\boldsymbol{\psi}(\mu_2)\mathbf{q}\mathbf{q}^T\boldsymbol{\epsilon}\boldsymbol{\psi}(\mu_2)^{-1}, \tag{3.27a}$$

$$D = \frac{a}{s(\mu_1 - \mu_2)}\frac{1}{m^2+n^2}\,[-nI + \mu_2{}^{-1}m\mathbf{g}\boldsymbol{\epsilon}]\boldsymbol{\psi}(\mu_2) \times \mathbf{q}\mathbf{q}^T\boldsymbol{\epsilon}\boldsymbol{\psi}(\mu_2)^{-1}\mathbf{g}\boldsymbol{\epsilon}, \tag{3.27b}$$

where

$$m = 1 - \frac{a\mu_2}{s(\mu_1 - \mu_2)}\,\mathbf{q}^T\boldsymbol{\epsilon}\boldsymbol{\psi}(\mu_2)^{-1}\dot{\boldsymbol{\psi}}(\mu_2)\mathbf{q}, \tag{3.28a}$$

$$n = -\frac{a}{s(\mu_1 - \mu_2)^2}\,\mathbf{q}^T\boldsymbol{\epsilon}\boldsymbol{\psi}(\mu_2)^{-1}\mathbf{g}\boldsymbol{\epsilon}\boldsymbol{\psi}(\mu_2)\mathbf{q}, \tag{3.28b}$$

$$\mathbf{q} = \begin{pmatrix} q^1 \\ q^2 \end{pmatrix}, \quad \mathbf{q}^T\boldsymbol{\epsilon} = (-q^2, q^1). \tag{3.29}$$

This completes the determination of $\boldsymbol{\psi}'(\lambda)$:

$$\boldsymbol{\psi}'(\lambda) = (\mathbf{b}^{-1})^T\left[I + \frac{C}{\lambda - \mu_2} + \frac{D}{\lambda - \mu_1}\right]\boldsymbol{\psi}(\lambda)u(t)^{-1}\mathbf{b}^{-1}. \tag{3.30}$$

Reversal of the trivial transformation represented by Eq. (3.23) yields[30]

$$\boldsymbol{\psi}'(\lambda) = \left[I + \frac{C}{\lambda - \mu_2} + \frac{D}{\lambda - \mu_1}\right]\boldsymbol{\psi}(\lambda)u(t)^{-1}. \tag{3.31}$$

Equation (3.20) for the transform of the complex $F(t)$ potential can now be obtained from Eq. (3.31) by a straightforward application of the formulas of Appendix B [see also Eqs. (6.15)–(6.37) of Ref. 16].

It is instructive to compare this pure-nonsoliton transformation with the pure-soliton "extended" HKX transformation[16] for which

$$\boldsymbol{\psi}'_{\rm ph}(\lambda) = \left[I + \frac{C}{\lambda - \mu_2} + \frac{D}{\lambda - \mu_1}\right]\boldsymbol{\psi}(\lambda), \tag{3.32}$$

and $G_0(t) = I = u(t)$. This is precisely the BZ two-soliton transformation in the limit where the two poles of $\boldsymbol{\chi}_2(\lambda)$ in Γ_2 coalesce to form a double pole at $\lambda = \mu_2$. The quantity in square brackets must now be interpreted as $\Delta^{-1/2}\boldsymbol{\chi}_2(\lambda)$ and so, from Eqs. (2.13b) and (2.22), we have

$$\begin{aligned}\boldsymbol{\chi}_1(\lambda) = \boldsymbol{\chi}_2(\lambda) &= \frac{\lambda - \mu_1}{\lambda - \mu_2}\left[I + \frac{C}{\lambda - \mu_2} + \frac{D}{\lambda - \mu_1}\right] \\ &= I + \frac{C + D + (\mu_2 - \mu_1)I}{\lambda - \mu_2} + \frac{(\mu_2 - \mu_1)C}{(\lambda - \mu_2)^2}.\end{aligned} \tag{3.33}$$

We see that the only difference between the pure-soliton and pure-nonsoliton versions of the null HKX transformation is the trivial transformation of Ref. 30 and a different choice of gauge. It is just the singularities of the gauge functions which are manifest in $\Delta^{-1/2}\boldsymbol{\chi}_2(\lambda)$ in Γ_2. The only significant differences appear in the statements of the respective HHP's and the actual details of the derivations: in the pure-nonsoliton case, C and D are uniquely determined from the HHP; in the pure-soliton case, a partial fraction expansion of the form (3.33) is the starting point and C and D are determined by Eq. (2.8) and the differential equations for the $\boldsymbol{\psi}(\lambda)$ potential,[2] aq^Aq^B being the constant of integration. More generally, a similar relationship can be established between the $2n$-fold Harrison transformation with $s = s_1, s_2, \ldots, s_{2n}$ (pure-nonsoliton) and the BZ $2n$-soliton transformation for which $\boldsymbol{\chi}_2(\lambda)$ has poles at $\lambda = \mu(s_1), \mu(s_2), \ldots, \mu(s_{2n})$ in Γ_2. However, $(2n-1)$-soliton BZ transformations contain as a factor the Kramer–Neugebauer involution[18] which is not in $\mathscr{K}$ but may be obtainable as some sort of fanciful limit of pure-nonsoliton transformations in $\mathscr{K}$—see Sec. 4.

The extended HKX transformation can also be derived from a slight modification of the HE formalism which closely parallels the BZ pure-soliton derivation. The representing matrix is

$$u_1(t) = g(t)u(t) = I, \tag{3.34}$$

but the contour L is drawn so that the pole in $u(t)$ and $g(t)$ at

$t = s$ is outside L, i.e., in L_-. The HHP takes the simple form

$$X_-(t) = X_+(t) = F'(t)F(t)^{-1}, \quad X_+(0) = I, \tag{3.35}$$

in which $X_+(t)$ is analytic in L_+ and $X_-(t)$ is analytic in L_- and at $t = \infty$ except for a simple pole at $t = s$. The general solution is

$$F'(t) = \left[I + \frac{stN}{t-s}\right]F(t), \tag{3.36}$$

where N is an arbitrary complex matrix function of (ρ, z). Equation (A7a) shows that N is traceless and degenerate, and Eq. (A7b) leads to a formula for N^* in terms of N. However, to completely determine N up to two constants of integration, it is necessary to substitute Eq. (3.36) into the differential equations (A1) and (A2). Not surprisingly, this is a somewhat lengthier derivation of Eq. (3.20), whether with $g(t) = u(t)^{-1}$ or $g(t) = I$, than the derivations above and in Ref. 11 where the HHP's had unique solutions.

C. Generation of the general Einstein–Rosen or Weyl solution from flat space

Kinnersley and Chitre[9] have shown that the abelian subgroup of **K** generated by the $\gamma_{12}^{(k)}$, $k = 0, 1, 2, \ldots$, maps flat space to the general Weyl solution

$$\mathscr{E} = f = e^{2\chi}, \quad \chi_{\rho\rho} + \rho^{-1}\chi_\rho + \chi_{zz} = 0, \tag{3.37}$$

for which $\chi(\rho, z)$ is analytic at and near (0, 0). More recently, Hauser and Ernst[4] generated the Weyl solution from flat space using their HHP with

$$u(t) = e^{\xi(t)\mathbf{k}}, \quad \mathbf{k} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.38}$$

where $\xi(t)$ is analytic in L_-, on L, and at $t = \infty$, and obtained the contour integral expressions, Eqs. (3.47a, b) below. In this subsection, we shall do the corresponding calculation in the BZ formalism and obtain the cylindrical-wave counterpart of the Weyl solution, the Einstein–Rosen solution.[21]

From Eqs. (2.35) and (A22), the BZ representing matrix is

$$G_0(t) = e^{2\xi(t)\mathbf{k}}, \tag{3.39}$$

where now $\xi(t) = \xi(t(\lambda))$ is analytic on and near Γ and at $\lambda = \lambda_{1,2}$. From Eq. (A20) and Appendix B, the $\psi(\lambda)$ potential for flat space is

$$\psi(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^2 + 2\beta\lambda + \alpha^2 \end{pmatrix}. \tag{3.40}$$

Since $\psi(\lambda)$ commutes with $G_0(t)$, it follows that $G(\lambda) = G_0(t)$. The BZ HHP (2.6a, b) can be reduced to a nonmatrix HHP by the substitutions,

$$\chi_1(\lambda) = \exp[2\eta_1(\lambda)\mathbf{k}], \quad \chi_2(\lambda) = \exp[2\eta_2(\lambda)\mathbf{k}], \tag{3.41}$$

where $\eta_{1,2}(\lambda)$ are nonmatrix functions of α, β, and λ analytic in $\Gamma + \Gamma_{1,2}$, respectively. The HHP takes the form

$$\eta_1(\lambda) = \eta_2(\lambda) + \xi(t(\lambda)), \quad \eta_1(\infty) = 0. \tag{3.42}$$

The unique solution is

$$\eta_{1,2}(\lambda) = -\frac{1}{2\pi i}\int_\Gamma \frac{\xi(t(\lambda'))}{\lambda' - \lambda}d\lambda', \quad \lambda \in \Gamma_{1,2}, \tag{3.43}$$

from which we obtain the transformed solution

$$\mathbf{g}' = \exp[2\eta_2(0)\mathbf{k}]\begin{pmatrix} 1 & 0 \\ 0 & \alpha^2 \end{pmatrix}, \tag{3.44a}$$

$$\psi'(\lambda) = \exp[2\eta_2(\lambda)\mathbf{k}]\begin{pmatrix} 1 & 0 \\ 0 & \lambda^2 + 2\beta\lambda + \alpha^2 \end{pmatrix}. \tag{3.44b}$$

This result is in agreement with Eqs. (3.37) and (A24) written in (α, β) coordinates with

$$\chi = \eta_2(0), \quad \beta(t) = 2\eta_2(\lambda) - \eta_2(0). \tag{3.45a,b}$$

Let us now deduce a contour integral for $\beta(t)$ in the cut complex t plane, the branch cut as usual joining the zeros of $S(t)$. First, from Eqs. (3.43) and (3.45b),

$$\begin{aligned} \beta(t) &= -\frac{1}{2\pi i}\int_\Gamma \left[\frac{2}{\lambda' - \lambda} - \frac{1}{\lambda'}\right]\xi(t(\lambda'))d\lambda' \\ &= -\frac{1}{2\pi i}\int_\Gamma \left[\frac{1}{\lambda' - \lambda} - \frac{1}{\lambda' - \alpha^2/\lambda}\right]\xi(t(\lambda'))d\lambda', \\ &\qquad \lambda \in \Gamma_2. \end{aligned} \tag{3.46}$$

Next, deform the circle Γ to a simple closed contour Γ' lying inside Γ such that the annular portion of Γ_2 between Γ' and Γ does not contain any singularities of $\xi(t(\lambda'))$ or the points $\lambda' = \lambda_2$ or $\lambda' = \lambda$. The change of variable $\lambda' = \mu(t')$ maps the contour Γ' to a negatively oriented (clockwise) contour L' in the cut t' plane which encloses the cut itself but no singularities of $\xi(t')$ nor the point $t' = t$. Since the integrand has no residue at $t' = \infty$, the contour L' can be replaced by a positively oriented contour L such that the cut lies in the exterior L_- and all the singularities of $\xi(t')$ and the point $t' = t$ lie in the interior L_+. The result of the transformation is

$$\chi = -\frac{1}{2\pi i}\int_L \frac{\xi(t')}{t'S(t')}dt', \tag{3.47a}$$

$$\beta(t) = -\frac{1}{2\pi i}\int_L \frac{S(t)}{S(t')}\frac{\xi(t')}{t' - t}dt', \quad t \in L_+, \tag{3.47b}$$

in exact agreement with Ref. 4. By differentiating under the integral sign, one can readily show that χ and $\beta(t)/S(t)$ satisfy the cylindrical wave equation and that Eq. (A25) in (α, β) coordinates holds.

D. Generation of the nonlinear superposition of n Kerr–NUT particles from n Schwarzschild particles using the B group

The **B** group of Kinnersley and Chitre[10] is represented by

$$u(t) = \begin{pmatrix} \cos\theta(t) & t^{-1}\sin\theta(t) \\ -t\sin\theta(t) & \cos\theta(t) \end{pmatrix}, \tag{3.48}$$

where $\theta(t)$ is analytic in $L + L_-$ and at $t = \infty$. It was shown by KC to map flat space to itself, Schwarzschild to Kerr–NUT, and the Zipoy–Voorhees solutions[34] with integer δ-parameter to generalizations of the Tomimatsu–Sato solutions,[35] to be called the KC solutions. Hauser and Ernst have generated Kerr from Schwarzschild using contour integration methods[3] but the same results can be obtained directly from the HE HHP. We shall show that, more generally, the nonlinear superposition of n Kerr–NUT particles ($4n$ parameters: mass, angular momentum, NUT parameter, and position on z axis for each particle) can be generated from the superposition of n Schwarzschild particles ($2n$ parameters) using the HE representation of the **B** group. This is not necessarily the best method of calculation: the general n-Kerr–NUT solution can be generated more easily from flat space by applying $2n$ Harrison transformations[16,28,36] or the BZ $2n$-soliton transformation.[2]

The HE HHP can provide a rational explanation of the fact that the infinite-dimensional **B** group generates only a finite number of arbitrary constants in these cases. First, the $F(t)$ potential of the input Weyl solution (n-Schwarzschild, Zipoy–Voorhees) must *not* be put in special HE gauge, but the unique gauge for which

$$\beta(t) = \beta_{\text{odd}}(t), \tag{3.49}$$

as discussed in Appendix A. [Equation (3.47b) gives $\beta_{\text{odd}}(t)$ when $t \in L_-$, special HE gauge when $t \in L_+$.] The t-plane singularities of $\beta_{\text{odd}}(t)$ all lie in L_-. Second, before specializing the harmonic function χ, a short calculation shows that

$$G(t) \equiv F(t) u(t) F(t)^{-1} \tag{3.50}$$

is even in $S(t)$ and is in fact analytic at the branch points of $S(t)$. The χ-potential for n Schwarzschild particles is the Newtonian potential of n rods with (mass)/(length) $= 1/2$ in geometric units and with endpoints at $z = z_i \pm \kappa_i$, $i = 1, 2, \ldots, n$, $\kappa_i > 0$, on the z axis. (A Zipoy–Voorhees particle is the limiting case where δ rods coincide.) The formulas for χ and $\beta_{\text{odd}}(t)$ are

$$\chi = \sum_{i=1}^{n} \tfrac{1}{2} \ln \frac{x_i - 1}{x_i + 1}, \tag{3.51}$$

$$\beta_{\text{odd}}(t) = \sum_{i=1}^{n} \tfrac{1}{2} \ln \frac{x_i(1 - 2tz_i) - 2\kappa_i t y_i - S(t)}{x_i(1 - 2tz_i) - 2\kappa_i t y_i + S(t)}, \tag{3.52}$$

where (x_i, y_i) are prolate spheroidal coordinates for each particle defined by

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} = (2\kappa_i)^{-1}[\rho^2 + (z - z_i + \kappa_i)^2]^{1/2} \pm (2\kappa_i)^{-1}[\rho^2 + (z - z_i - \kappa_i)^2]^{1/2}, \tag{3.53}$$

positive for x_i, negative for y_i. The key property is that $G(t)$ and $G(t)^{-1}$ are analytic throughout L_- except for simple poles at

$$t = \tfrac{1}{2}(z_i \pm \kappa_i)^{-1} \equiv s_1, s_2, \ldots, s_{2n}, \tag{3.54}$$

when these numbers are distinct, and poles of higher multiplicity when there are coincidences.

From Eqs. (1.14) and (3.50),

$$X_+(t) = X_-(t) G(t)^{-1}, \tag{3.55}$$

of which the left-hand side is analytic in L_+ and the right-hand side is analytic in L_- except for the aforementioned poles at $t = s_1, \ldots, s_{2n}$. It follows that $X_+(t)$ is a rational function of t of the form, when the s_j are distinct,

$$X_+(t) = F'(t) F(t)^{-1} = I + \sum_{j=1}^{2n} \frac{tA_j}{t - s_j}, \tag{3.56}$$

the A_j being complex matrix functions of (ρ, z) to be determined. When δ of the s_j are equal, a sum of partial fractions appropriate to a pole of order δ should be included in $X_+(t)$. Finally, the requirement of analyticity of $X_-(t) = X_+(t) G(t)$ at the points $t = s_1, \ldots, s_{2n}$ in L_- gives $4n$ column-vector equations for the $2n$ unknown matrices A_j, uniquely determining the latter. The final solution does not depend on the detailed behavior of the function $\theta(t)$ but only on the values of $\theta(t)$ at $t = s_1, \ldots, s_{2n}$ (and derivatives of orders up to $\delta - 1$ when δ of the s_j are equal).

The above discussion will assist us in determining how the **B** group is represented in the BZ formalism. We shall remain in the context of stationary axisymmetric fields. First, since $h(t)$ is given by Eq. (A22) for all Weyl solutions when the gauge obeys Eq. (3.49), it follows from Eqs. (2.35) and (3.48) that

$$G_0(t) = I, \quad \chi_1(\lambda) = \chi_2(\lambda). \tag{3.57}$$

Clearly, the transformation is of *pure-soliton* type. Since the function $\theta(t)$ has disappeared from the problem, the arbitrary constants generated by the transformation will not appear now as values of $\theta(t)$ at $t = s_1, \ldots, s_{2n}$ but as integration constants. From Eqs. (2.13b), (2.40a), and (3.56), we see that $\Delta^{-1/2}\chi_2(\lambda)$ has poles at

$$\begin{aligned} \lambda &= \mu(s_1), \ldots, \mu(s_{2n}) \quad \text{in } \Gamma_2, \\ \lambda &= \nu(s_1), \ldots, \nu(s_{2n}) \quad \text{in } \Gamma_1, \end{aligned} \tag{3.58}$$

simple if the s_j are distinct, nonsimple otherwise. But $\chi_2(\lambda)$ itself cannot have simple poles in Γ_2 for then $\Delta^{-1/2}\chi_2(\lambda)$ would have quadratic branch points according to Eq. (2.22). Thus $\chi_2(\lambda)$ has *double* poles at $\lambda = \mu(s_1), \ldots, \mu(s_{2n})$ in Γ_2 when the s_j are distinct, and poles of double the previous order otherwise. It then follows that $\chi_2(\lambda)$ is analytic throughout Γ_1.

The preceding paragraph identifies the n-Schwarzschild to n-Kerr–NUT transformation as a special case of the $4n$-*soliton transformation* in which the poles of $\chi_2(\lambda)$ in Γ_2 are all of even order. When the s_j are distinct, the full transformation factorizes into $2n$ extended HKX transformations,[16] whose q^A parameters must be chosen appropriately. For example, two of these special extended HKX transformations map Schwarzschild to Kerr–NUT, four map double Schwarzschild to double Kerr–NUT,[37] two of rank zero and two of rank one[12,38] map Zipoy–Voorhees $\delta = 2$ to the full KC $\delta = 2$ solution.[10]

Only $2n$ of the $4n$ integration constants in the $4n$-soliton transformation can be chosen independently as there are only $2n$ values of $\theta(s_j)$. Also, since the n-Schwarzschild solution is the general *static* $2n$-soliton solution (meaning $2n$-soliton transform of flat space), the general $4n$-soliton transform of the n-Schwarzschild solution with same s-parameters will be a special limiting form of the $6n$-soliton solution or $3n$-Kerr–NUT solution in which the $6n$ s-parameters coincide in threes. This latter solution may be interpreted as the nonlinear superposition of n KC $\delta = 3$ particles in which each particle has six of its eight allowed degrees of freedom (including NUT and z position). We need to reduce the six degrees of freedom to four in such a way that each KC $\delta = 3$ particle reduces to just a Kerr–NUT particle.

It was claimed in Ref. 16 and will now be proven that n-Schwarzschild transforms into n-Kerr–NUT under $2n$ of the special extended HKX transformations for which

$$q^A = (1, 0) \quad \text{(type I)}, \tag{3.59a}$$

$$\text{or} \quad q^A = (0, 1) \quad \text{(type II)}. \tag{3.59b}$$

(The type I transformations are precisely the original HKX transformations of Ref. 12 and type II can be obtained from type I by interchanging the tensor indices 1 and 2.) A consequence of Eq. (6.46) of Ref. 16 is that these special HKX transformations factorize into two BZ one-soliton transformations of which the first maps static solutions to static and the second maps static to stationary. Let the static-to-static BZ transformations be denoted type I or type II according to the type of the HKX transformations. Since BZ pure-soliton transformations commute exactly, we may perform the $2n$ static-to-static transformations first, choosing the type so as to cancel the Schwarzschild particles one by one and eventually leave flat space. Then the remaining $2n$ static-to-stationary transformations will map flat space to the nonlinear superposition of n Kerr–NUT particles.

The explicit transform of the general Weyl solution under the type I and type II BZ transformations is easily calculated from Eq. (5.19) of Ref. 16. The results are

$$\text{type I BZ:} \quad e^{2\chi'} = \alpha^{-1}\mu(s)e^{2\chi}, \tag{3.60a}$$

$$\text{type II BZ:} \quad e^{2\chi'} = \alpha^{-1}\nu(s)e^{2\chi}, \tag{3.60b}$$

where $\alpha = i\rho$. Consider the case where the Schwarzschild rod singularities are nonoverlapping and take them one at a time. Suppose a given rod connects $z = z_0 - \kappa = (2s_1)^{-1}$ to $z = z_0 + \kappa = (2s_2)^{-1}$ along the z axis, with $\kappa > 0$. Three cases need to be distinguished:

Case 1: $0 < (2s_1)^{-1} < (2s_2)^{-1}$;

Case 2: $(2s_1)^{-1} < 0 < (2s_2)^{-1}$;

Case 3: $(2s_1)^{-1} < (2s_2)^{-1} < 0$.

The correct choice of BZ or HKX type depends on the value of (ρ, z) and the location of the branch cut joining the zeros of $S(t)$ in the t plane. To remove this ambiguity, let the cut intersect the real axis at points $t < \min(s_1,...,s_{2n})$ or $t > \max(s_1,...,s_{2n})$ so that $S(s_j) > 0$ for all j. A straightforward calculation now shows that the selected Schwarzschild rod singularity can be cancelled out by applying the following types of double BZ transformation:

Case 1: type I at $s = s_1$ and type II at $s = s_2$;

Case 2: type II at $s = s_1$ and $s = s_2$;

Case 3: type II at $s = s_1$ and type I at $s = s_2$,

$s = (2z)^{-1}$. The corresponding types of double HKX transformation map the Schwarzschild particle to Kerr–NUT as required rather than to KC $\delta = 3$. The applications of the **B** group in Refs. 3 and 10 belong to *Case 2*. Additional cases can be introduced by changing the sign of some of the x_i in Eq. (3.51) (negative mass particles) and/or replacing some of the x_i by $\pm y_i$.

4. LIMITING TRANSITIONS AND RELAXATION OF BOUNDARY CONDITIONS

We have already remarked that the boundary conditions at $t = \infty$ chosen by Hauser and Ernst are not necessary for the preservation of Einstein's equations, but have the convenient property of guaranteeing uniqueness of the solution of the HHP. For example, there are many transformations in $\mathscr{K}$ represented by $u(t)$ matrices with some sort of singularity at $t = s$, say, in L_+, which have well-defined limits as $s \to \infty$. Simple examples which can be treated by elementary methods are the $s = \infty$ limits of the Harrison and HKX transformations (and products thereof) for which $u(t)$ and/or $X_-(t)$ have, respectively, quadratic branch points and poles at $t = \infty$. Here, we shall discuss the $s = \infty$ null HKX transformation, which preserves asymptotic flatness as for finite s.

A formula for the $s = \infty$ null HKX transform of $F(t)$ can be obtained by simply putting $s = \infty$ in Eq. (3.20). (Actually, in Ref. 16 it was shown by direct exponentiation that the $s = \infty$ *nonnull* HKX transformation is given by essentially the same expression.[39]) The formula can be substantially simplified by observing that the limiting forms of Eqs. (A1) and (A7a,b) imply that

$$F_{AB}(\infty) = F_A h_B, \tag{4.1}$$

where F_A is a complex vector function of (ρ, z) or (α, β) and h_B is a *real constant* vector, in any gauge for which $F_{AB}(t)$ is analytic at $t = \infty$. Similarly, the generating function $G_{AB}(s, t)$ defined by Eq. (3.21) is well defined as either s or t or both tend to infinity, the limiting forms being

$$G_{AB}(s, \infty) = -S(s)F_{XA}(s)F^X h_B, \tag{4.2a}$$

$$G_{AB}(\infty, t) = \epsilon_{AB} + 2trF_X F^X{}_B(t)h_A, \tag{4.2b}$$

$$G_{AB}(\infty, \infty) = \epsilon_{AB} + G_{BA}(\infty, \infty) = g_A h_B + G h_A h_B, \tag{4.2c}$$

where $r = (\rho^2 + z^2)^{1/2} = (\beta^2 - \alpha^2)^{1/2}$, G is a complex scalar function of (ρ, z) or (α, β), and g_A is any real constant vector satisfying $g_X h^X = 1$ (i.e., $g_1 h_2 - g_2 h_1 = 1$). [We choose a (ρ, z) domain and branch cut so that $S(t)/t \to 2r$ as $t \to \infty$: if the cut is a straight line segment, then $z < 0$ or $\beta < -\alpha$.] From Eqs. (A1) and (A7b), the vector F_A and scalar G satisfy

$$\nabla F_A = -(i/2r^2)[z\nabla H_{AX} + \rho\tilde{\nabla} H_{AX}]F^X, \tag{4.3a}$$

$$\nabla G = F^*{}_X \nabla F^X = (i/2r^3)[(z^2 - \rho^2)\nabla H_{XY} + 2\rho z\tilde{\nabla} H_{XY}]F^X F^Y, \tag{4.3b}$$

$$F^*{}_A = r^{-1}(zF_A + if_{AX}F^X), \tag{4.4a}$$

$$G^* = G - ir^{-1}f_{XY}F^X F^Y. \tag{4.4b}$$

With the aid of Eqs. (4.1) and (4.2b,c), the limiting form of Eq. (3.20) reduces to

$$F'_{AB}(t) = F_{AB}(t) + 2ktrF_A F_X F^X{}_B(t)/(1 - kG), \tag{4.5}$$

where $k = a(q^X h_X)^2/(1 - aq^X q^Y g_X h_Y)$; k is the only essential parameter. Transformations of the form (4.5) form a one-parameter Lie group and hence iteration does not lead to further transformations.

It is not difficult to deduce from Eq. (4.5) an HHP which admits Eq. (4.5) as a solution. It is more instructive, of course, to see the HHP derived from the limit as $s \to \infty$ of the HHP for the finite-s HKX transformation. For this purpose the form (3.35) of the HHP in which the pole at $t = s$ is *outside* the contour L is more useful. The limiting form of Eq. (3.35) as $s \to \infty$ is

$$X_-(t) = X_+(t) = F'(t)F(t)^{-1}, \quad X_+(0) = I, \tag{4.6}$$

where $X_+(t)$ is analytic in $L + L_+$, $X_-(t)$ is analytic in $L + L_-$, and

$$X_-(t) = O(t) \text{ as } t \to \infty. \tag{4.7}$$

The $u(t)$ matrix is the unit matrix I. The general solution of this HHP is

$$F'_{AB}(t) = (\epsilon_A{}^X - tN_A{}^X)F_{XB}(t), \tag{4.8}$$

where $N_A{}^B$ is an arbitrary complex function of (ρ, z). To identify $N_A{}^B$, it is necessary to go back to the differential and algebraic relations satisfied by $F'(t)$. Equation (A7a) shows that $N_A{}^B$ is null (i.e., $\det N = 0$) and traceless (i.e., $N_{AB} = N_{BA}$) and so we have the factorization, $N_A{}^B = N_A N^B$. From Eqs. (4.1) and (4.8), the condition that $F'_{AB}(t)$ be analytic at $t = \infty$ requires that $N^X F_X = 0$, implying that N_A is proportional to F_A. However, this condition can be relaxed as an $O(t)$ term in $F'_{AB}(t)$ can be absorbed by a change of gauge.

The differential equation (A1) for $F'(t)$ implies that the vector N_A satisfies

$$2z\nabla N_A - 2\rho\tilde{\nabla}N_A = -iN^X\nabla H_{AX} - \tfrac{1}{2}N_A N^X \nabla N_X + N_A \nabla z. \tag{4.9}$$

This equation can be solved by first looking for a particular integral proportional to F_A and comparing with Eqs. (4.3a,b). The particular integral is found to be

$$N_A = [2kr/(1-kG)]^{1/2}F_A, \tag{4.10}$$

where k is a real constant. Substitution of Eq. (4.10) into (4.8) gives the $s = \infty$ HKX transformation (4.5).

This transformation can also be obtained from the pure-nonsoliton part of the BZ formalism by relaxing conditions (2.4) and (2.5). If conditions (2.4) and (2.5) are replaced by the requirements that $\psi(\lambda)$ and $G_0(t)$ be analytic at $\lambda = \lambda_2$, the solution of the BZ HHP remains unique and the resulting transformations are products of elements of $\mathscr{K}$, the $s = \infty$ HKX transformation (4.5), and the unimodular linear transformation of Killing vectors. Consider the BZ HHP in the form (2.37a,b) with

$$u(t) = \mathbf{d} \quad \text{or} \quad u^A{}_B(t) = d^A{}_B, \tag{4.11}$$

$\mathbf{d} = d^A{}_B$ a constant matrix, $\det \mathbf{d} = 1$, and let $\psi(\lambda)$ be in modified special HE gauge,[30] i.e., $\psi(\lambda)$ is analytic in $\Gamma + \Gamma_2$ and $\psi_{AB}(\lambda_2) = 2r(\mathrm{Re}\, F_A)h_B$. The solution of the HE HHP with $u^A{}_B(t) = d^A{}_B$ is simply a rotation of Killing vectors[11]:

$$f'_{AB} = d_A{}^X d_B{}^Y f_{XY}, \quad F'_{AB}(t) = d_A{}^X d_B{}^Y F_{XY}(t) \tag{4.12}$$

(note $d_A{}^B = \boldsymbol{\epsilon}\, \mathbf{d}\, \boldsymbol{\epsilon}$). The method of solution of the BZ HHP follows almost exactly the same lines as in Sec. 3B, so it is not necessary to show details. The final result is found to be

$$F'_{AB}(t) = d_A{}^X d_B{}^Y\left[F_{XY}(t) - \frac{2ntrF_X F_Z F^Z{}_Y(t)}{m+nG}\right], \tag{4.13}$$

where

$$m = d^X{}_Y h_X g^Y, \quad n = d^X{}_Y h_X h^Y. \tag{4.14}$$

The transformation (4.13) is clearly the product of the transformations (4.5) with $k = -n/m$ and (4.12). The factor (4.12) can be chosen to be the simple translation of Footnote 30, but not the identity transformation for then n would be zero.

The general solution of Eq. (4.9) leads to a two-parameter enlargement of the $s = \infty$ HKX transformation. First, by considering gauge changes in $F(t)$ with $g(t) = O(t)$ as $t \to \infty$, the general solution of Eq. (4.3a) can be shown to be

$$F_A^{(\mathrm{gen})} = aF_A + bF^{(1)}{}_A{}^X h_X, \tag{4.15a}$$

a, b real constants, where $F^{(1)}_{AB}$, $F^{(2)}_{AB}$,..., are the coefficients in the descending power series,

$$F_{AB}(t) = F_A h_B + t^{-1}F^{(1)}_{AB} + t^{-2}F^{(2)}_{AB} + \cdots,$$

which converges for $|t| > (2r)^{-1}$ in modified special HE gauge. [Equations (A1) and (A7a,b) lead to differential and algebraic equations for the $F^{(n)}_{AB}$ and Eqs. (4.2a,b,c) imply $2rF^X F^{(1)}_{XB} = -g_B - Gh_B$.] Substituting $F_A^{(\mathrm{gen})}$ for F_A in the right-hand side of Eq. (4.3b) and integrating gives

$$G^{(\mathrm{gen})} = a^2G + 2abr[F_X h_Y F^{(2)XY} - \det F^{(1)}] + 2b^2 r h_X h_Y F^{(1)}{}_Z{}^X F^{(2)ZY}. \tag{4.15b}$$

The general solution of Eq. (4.9) is now

$$N_A = \left(\frac{2r}{1 - G^{(\mathrm{gen})}}\right)^{1/2} F_A^{(\mathrm{gen})}, \tag{4.16}$$

and the enlarged $s = \infty$ HKX transformation takes the form

$$F'_{AB}(t) = F_{AB}(t) + \frac{2trF_A^{(\mathrm{gen})}F_X^{(\mathrm{gen})}F^X{}_B(t)}{1 - G^{(\mathrm{gen})}}, \tag{4.17}$$

reducing to (4.5) when $a^2 = k$, $b = 0$. This transformation preserves asymptotic flatness and, in particular, maps flat space to extreme Kerr–NUT.

The transformation (4.17) cannot be iterated as it stands because $F'(t) = O(t)$ as $t \to \infty$. This can be repaired by the gauge change,

$$F''_{AB}(t) = F'_A{}^X(t)g_{XB}(t), \quad g_{AB}(t) = \epsilon_{AB} + (b/a)th_A h_B. \tag{4.18}$$

The limit as $t \to \infty$ now gives

$$F''_A = \frac{a^{-1}F_A^{(\mathrm{gen})}}{1 - G^{(\mathrm{gen})}}, \quad G'' = \frac{a^{-2}G^{(\mathrm{gen})}}{1 - G^{(\mathrm{gen})}}, \tag{4.19a,b}$$

$$h''_A = h_A + a^2 g_A, \quad g''_A = g_A. \tag{4.19c,d}$$

The combined transformation given by Eqs. (4.17) and (4.18) satisfies the HHP,

$$X_-(t) = F''(t)u(t)F(t)^{-1}, \tag{4.20}$$

$X_-(t)$ analytic in $L + L_-$, $F''(t)$ analytic in $L + L_+$, with

$$u(t) = u^A{}_B(t) = -\epsilon^A{}_B + (b/a)th^A h_B, \tag{4.21}$$

and boundary conditions

$$X_-(t) = O(t), \quad F''(t) = O(1) \text{ as } t \to \infty, \tag{4.22}$$

and, as usual, $F''_{AB}(0) = i\epsilon_{AB}$. These transformations can be iterated and do not close to form a finite-dimensional Lie group but span an infinite-dimensional subgroup of $\mathbf{K}$ outside $\mathscr{K}$ which preserves asymptotic flatness. The product of n such transformations satisfies an HHP of the form (4.20) with

$$u(t) = O(t^n), \quad X_-(t) = O(t^n), \quad F''(t) = O(1) \tag{4.23}$$

as $t \to \infty$, and with $u(t)$ a matrix polynomial in t of degree n. If

$u(t)$ also has singularities in L_+, then the solution of the HHP is the product of an element of $\mathscr{K}$ and n transformations (4.17)–(4.18).

The Harrison transformation also has a well-defined limit as $s \to \infty$, namely

$$F'(t) = \begin{pmatrix} T & -1 \\ -t^{-1} - i\mathscr{E}T & i\mathscr{E} \end{pmatrix} F(t) \begin{pmatrix} -(h_2/h_1)t & 1 \\ t & 0 \end{pmatrix}, \tag{4.24a}$$

where $T = F_2/F_1$, and can be enlarged to have one continuous parameter (b/a) by taking

$$T = F_2^{(\mathrm{gen})}/F_1^{(\mathrm{gen})}. \tag{4.24b}$$

It is a trivial matter to write down an HHP for this transformation in which

$$u(t) = O(t^{1/2}), \quad X_-(t) = O(t^{1/2}), \tag{4.25}$$

as $t \to \infty$, the branch cut joining $t = 0$ to $t = \infty$. In Eq. (4.24a), we have deliberately chosen a gauge such that the $s = \infty$ Harrison transformation can be iterated. The product of two of the transformations (4.24) is identical to the transformation (4.17)–(4.18) up to a translation $\omega \to \omega + \text{constant}$.[30]

There is an important discrete involution due to Kramer and Neugebauer[18] which plays a major role in the theory of the Geroch group and Bäcklund transformations[16,28] but which lays outside all existing representations of $\mathbf{K}$. This transformation, to be denoted (I), has the effect,

$$f' = \rho f^{-1}, \quad \omega' = j\psi, \tag{4.26a,b}$$

$$\psi' = -j\omega, \quad \mathscr{E}' = \rho f^{-1} - ij\omega, \tag{4.26c,d}$$

where $j = \sqrt{(-1)}$ and the symbols are defined by Eqs. (1.4) and (1.8). [The complex conjugation operation of Eq. (A7b) does not apply to j, i.e., $i^* = -i, j^* = j$.] A natural question to ask is whether (I) can be obtained as some suitably defined limit of a sequence of elements of $\mathscr{K}$ as in the preceding paragraphs. Such a limit, if it exists, would be rather contrived and probably not have much practical utility. We shall conclude this paper by presenting an HHP for (I) and a one-parameter family of limiting transitions which work for Weyl solutions but not for stationary solutions.

In Ref. 16, the (I)-transform of $F(t)$ was calculated by first calculating the transforms of $F(t)$ under $\mathbf{Q}$[14,16,40] and $\tilde{\mathbf{Q}} = (I)\mathbf{Q}(I)$ with the aid of Neugebauer's commutation theorem[28] and the KC recurrence relations.[8] Using Eq. (A7b), the result can be written,

$$F'(t) = (1 - 2tz + 2jt\rho)^{-1/2}\left[jt^{-1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + A\right] \times F(t)\begin{pmatrix} 0 & 1 \\ jt & 0 \end{pmatrix}, \tag{4.27a}$$

where

$$A = \begin{pmatrix} ij\mathscr{E}' & -1 \\ \mathscr{E}\mathscr{E}' - 2(\rho + jz) & -ij\mathscr{E} \end{pmatrix} \tag{4.27b}$$

and $\mathscr{E}'$ is given by Eq. (4.26d). The (I)-transform of $F^*(t)$ can be deduced by taking the i-complex conjugate of both sides of Eqs. (4.27a,b). If $F(t)$ is in special HE gauge, then $F'(t)$ is analytic in the whole complex t plane except for quadratic branch points of index $-\frac{1}{2}$ at the two zeros of $S(t)$ *as well as at* $t = \infty$, necessitating two cuts. This is an optimum choice of gauge in this case as one of the conditions for the existence of special HE gauge [namely, $f' \neq 0$ on z axis near (0, 0)] is not met; in fact $f'(\rho, z) = O(\rho)$ as $\rho \to 0$ for all points on the axis at which the original solution is nonsingular.

The right-hand side of

$$X_-(t) = (1 - 2tz + 2jt\rho)^{1/2} F'(t)\begin{pmatrix} 0 & -jt^{-1} \\ 1 & 0 \end{pmatrix} F(t)^{-1} \tag{4.28}$$

is analytic in L_- and at $t = \infty$ and so Eq. (4.28) may be regarded as an HHP for the unknowns $X_-(t)$ and $F'(t)$. (It can be interpreted in one complex dimension by writing $j = -\epsilon i, \epsilon = \pm 1$, and later replacing ϵ by ij.) With the boundary conditions (A6a,b) on $F'(t)$ supplemented by either Eq. (A7a) or the relation $H'_{12} - H'_{21} = 2iz$, the solution (4.27) is unique except that $\mathscr{E}'$ is an undetermined function of (ρ, z). Equation (4.26d) for $\mathscr{E}'$ can then be deduced from either Eq. (A2) or (A7b).

Equation (4.28) suggests that a suitable candidate for the representing matrix of the (I) transformation is

$$u_0(t) = (-jt)^{1/2}\begin{pmatrix} 0 & -jt^{-1} \\ 1 & 0 \end{pmatrix}. \tag{4.29}$$

This conclusion is reinforced by the fact that if an element $g \in \mathbf{K}$ is represented by $u(t)$ and its dual element $\tilde{g} = (I)g(I)$ by $\tilde{u}(t)$, then

$$\tilde{u}(t) = u_0(t)u(t)u_0(t). \tag{4.30}$$

Equation (4.30) can be proved without difficulty from Eqs. (3.19a,b,c) of Ref. 16 which relate the infinitesimal $\gamma_{AB}^{(k)}$ transformations[8] to their duals and Eqs. (4.4)–(4.6) of Ref. 11 which relate the $\gamma_{AB}^{(k)}$ to their representing matrices. Furthermore, the representing matrix of the BZ one-soliton transformation can be calculated by multiplying the $u(t)$ matrices of the respective factors in Eq. (5.28) of Ref. 16 [Harrison transformation; (I) transformation; translation, scaling, and gauge transformations] and the result is, perhaps not surprisingly, the unit matrix.

Let (J) denote the product of the involution $\mathscr{E} \to \mathscr{E}^{-1}$ (gravitational duality rotation through π radians) and the (I) transformation. Since (J) has the effect, $e^{2\chi'} = \rho e^{2\chi}$, on the general Weyl solution, $\mathscr{E} = e^{2\chi}$, it follows that if $(J) \in \mathbf{K}$, then (J) must be in some analytic extension of the subgroup of $\mathscr{K}$ represented by Eq. (3.38). The effect of the subgroup (3.38) on Weyl solutions is the linear superposition,

$$e^{2\chi'} = e^{2\bar{\chi} + 2\chi}, \quad e^{2\beta'(t)} = e^{2\bar{\beta}(t) + 2\beta(t)}, \tag{4.31}$$

where $\bar{\chi}$ and $\bar{\beta}(t)$ are given by the right-hand sides of Eqs. (3.47a,b) with $\xi(t) = -\bar{\chi}(0, (2t)^{-1})$; note that $e^{2\bar{\chi}(\rho, z)}$ is analytic and nonvanishing at and near the origin. The (J) transformation represents a linear superposition of χ with the potential of an infinite cylindrical bar of line density $\frac{1}{4}$ (recall that the Schwarzschild "rod" has line density $\frac{1}{2}$). If a piece of the bar containing the origin is removed, say $z_0 - \kappa < z < z_0 + \kappa$, then the superposition is of the form (4.31) with

$$e^{2\bar{\chi}} = \kappa(x + 1)(1 - y^2)^{1/2}, \tag{4.32}$$

where the (x, y) are prolate spheroidal coordinates [see Eq.

(3.53) with $z_i = z_0$ and subscript i dropped] and is represented by

$$u(t) = \begin{pmatrix} \sigma^{-1}(t) & 0 \\ 0 & \sigma(t) \end{pmatrix}, \quad \sigma(t) = [4\kappa^2 - t^{-2}(1 - 2tz_0)^2]^{1/4}. \tag{4.33}$$

Let this transformation be denoted (J_κ); $(J_\kappa) \in \mathscr{K}$ when $\kappa > |z_0|$. Now fix (ρ, z) and z_0 and let κ decrease to $|z_0|$ and thence to zero: we find $e^{2\bar{\chi}} \to \rho$ and so $(J_\kappa) \to (J)$ as $\kappa \to 0$ when applied to Weyl solutions.

The (J_κ)-transform of a generic stationary solution also has a well-defined unambiguous limit as $\kappa \to 0$. The limit, however, depends on z_0 and is *different* from the (J)-transform for all z_0. To prove this statement, it is sufficient to exhibit a single counterexample. Now the (J_κ)-transform of a slowly rotating solution,

$$\mathscr{E} = e^{2\chi} + i\epsilon\psi + O(\epsilon^2), \tag{4.34a}$$

$$\nabla_3{}^2\chi = 0, \quad \nabla_3{}^2\psi - 4\nabla\chi\cdot\nabla\psi = 0, \tag{4.34b}$$

ϵ small, can be calculated explicitly by contour integration. First, contour integral expressions for $\psi(\rho, z)$ and, more generally, $F(\rho, z, t)$ analogous to Eqs. (3.47a,b) can be written in terms of $\chi(0, z)$ and $\psi(0, z)$, assumed to be analytic at and near $z = 0$. Equation (2.33) provides an infinity of $u(t)$ matrices of the form $u(t) = I + \epsilon v(t)$ which map $\mathscr{E}_{\text{Weyl}} = e^{2\chi}$ to $\mathscr{E} = e^{2\chi} + i\epsilon\psi$. Then, to first order in ϵ, the multiplicative HHP (1.14) reduces to an additive boundary value problem solvable by standard methods. The final results are

$$F(t) = [I + \epsilon Y_+(t)]F_{\text{Weyl}}(t), \tag{4.35a}$$

$$H = H_{\text{Weyl}} + i\epsilon\dot{Y}_+(0)\epsilon, \tag{4.35b}$$

$$\psi = -\dot{Y}_+(0)_{12}, \tag{4.35c}$$

where

$$Y_+(t) = -\frac{1}{2\pi i}\int_L \frac{t}{s-t}\,\frac{\psi(0, (2s)^{-1})e^{-2\beta(s)}}{S(s)} \times \begin{pmatrix} i\mu(s) & e^{2\chi} \\ \mu^2(s)e^{-2\chi} & -i\mu(s) \end{pmatrix} ds \tag{4.36}$$

for t in L_+. In these equations, $F_{\text{Weyl}}(t)$ is given by Eq. (A24), H and H_{Weyl} by Eq. (A6b), χ and $\beta(t)$ by Eqs. (3.47a,b), $\mu(t)$ by Eq. (B3a), and the contour L encloses all the singularities of $\psi(0, (2s)^{-1})$.

The (J_κ)-transform of the slowly rotating solution (4.34) is also given by Eqs. (4.35) and (4.36) with $\chi, \beta(t), \psi$, etc., replaced by primed variables, $\chi', \beta'(t), \psi'$, etc., calculated from Eqs. (4.31), (4.32), and

$$\psi'(0, (2s)^{-1}) = \sigma^2(s)\psi(0, (2s)^{-1}). \tag{4.37}$$

When $\kappa > |z_0|$, the contour L encloses the two branch points of $\sigma^2(s)$ at $s = \frac{1}{2}(z_0 \pm \kappa)^{-1}$ and the cut joining them, to be denoted C_1 (without loss of generality, take $z_0 > 0$). The cut C_2, joining the zeros of $S(s)$, crosses the real axis on either side of C_1. To understand how $Y'_+(t)$ varies as κ decreases to z_0 and beyond, view the complex s plane in the Riemann sphere topology. When $\kappa < z_0$, the cut C_1 occupies the two semi-infinite portions, $-\infty < s \leqslant \frac{1}{2}(z_0 + \kappa)^{-1}$, $\frac{1}{2}(z_0 - \kappa)^{-1} \leqslant s < \infty$, of the real axis, and C_2 passes between. The contour L takes the form of two open infinite arcs enclosing the two parts of C_1 in the positive sense, as well as the point t and all the singularities of $\psi(0, (2s)^{-1})$, and may be deformed to a clockwise contour L' enclosing only the cut C_2 (as in Sec. 3C). As $\kappa \to 0$, L (or L') and C_2 are pinched off at $s = (2z_0)^{-1}$ as the two ends of C_1 close up the gap, and the integral takes the limiting form

$$Y'_+(t) = \frac{1}{2\pi}\int_{L''} \frac{t}{s-t}\,\frac{\psi(0, (2s)^{-1})e^{-2\beta(s)}}{S(s)} \times \begin{pmatrix} -i\rho & \nu(s)e^{2\chi} \\ -\mu(s)e^{-2\chi} & i\rho \end{pmatrix} ds, \tag{4.38}$$

where L'' is a "figure-eight" shaped contour which crosses itself at $s = (2z_0)^{-1}$ and encloses C_2, $\operatorname{Im} s > 0$, in the positive direction and C_2, $\operatorname{Im} s < 0$, in the negative direction.

As an application of Eq. (4.38), the simple solution

$$\mathscr{E} = 1 + i\psi_0, \quad \omega = 0, \tag{4.39}$$

ψ_0 small constant, transforms under (J_κ) in the limit as $\kappa \to 0$ to

$$\mathscr{E}' = \rho + 2i\pi^{-1}\psi_0 r, \quad \omega' = \pi^{-1}\psi_0 \ln\frac{1 - \cos\theta}{1 + \cos\theta}, \tag{4.40}$$

to order ψ_0, where the spherical coordinates (r, θ) are defined by $\rho = r\sin\theta$, $z - z_0 = r\cos\theta$. For each z_0, this is different from the (J)-transform, which is simply $\mathscr{E}' = 1, \omega' = -j\psi_0$. Equations (4.38) and (4.40) also hold when $z_0 \leqslant 0$ [in the case $z_0 = 0$, the cut C_2 joins $s = \frac{1}{2}(z - i\rho)^{-1}$ to $i\infty$ and $\frac{1}{2}(z + i\rho)^{-1}$ to $-i\infty$ and L'' consists of two infinite lines crossing the plane from left to right, one enclosing C_2, $\operatorname{Im} s > 0$, in the positive sense, the other enclosing C_2, $\operatorname{Im} s < 0$, in the negative sense]. The limits as $z_0 \to \pm\infty$ are also well defined if one compensates unbounded terms by additive constants in ω' and ψ' and by suitably adjusting the gauge of $F'(t)$. Under this latter limit, the transform of (4.39) becomes

$$\mathscr{E}' = \rho \mp 2i\pi^{-1}\psi_0 z, \quad \omega' = \mp 2\pi^{-1}\psi_0 \ln\rho. \tag{4.41}$$

A somewhat different limiting transition can be carried out by superposing a finite rod of length 2κ and line density $\frac{1}{4}$, $e^{2\bar{\chi}} = 2\kappa[(x-1)/(x+1)]^{1/2}$ (Zipoy–Voorhees $\delta = \frac{1}{2}$ particle), and letting $\kappa \to \infty$ with the midpoint fixed at $z = z_0$. In this case, the limiting procedure involves rather lengthy and difficult asymptotic techniques and it is necessary to adjust gauge and admit the translation $\omega \to \omega +$ constant in order to absorb terms of order $\ln\kappa$ in $F'(t)$. The limiting transform of the solution (4.39) turns out to be (4.41) (upper sign if $z_0 < 0$, lower if $z_0 > 0$).

The limiting transitions discussed in the preceding paragraphs do not succeed in obtaining the Kramer–Neugebauer involution (I) as the limit of some sequence of members of $\mathscr{K}$. At this stage, the question of whether or not (I) is in some analytic continuation of $\mathscr{K}$ (and therefore in **K** by definition) remains open. Nevertheless, the foregoing calculations are instructive in that they show for the first time how large classes of solutions involving logarithmic singularities along the entire z axis can be generated from flat space using the Geroch group.

ACKNOWLEDGMENTS

This research was supported in part by the National Science Foundation (AST79-22012-A1).

APPENDIX A: ANALYTIC CONTINUATION OF $F(t)$ ONTO THE SECOND RIEMANN SHEET

The complex matrix potential $F(t)$ $[=F_{AB}(t)=F_{AB}(\rho,z,t)]$ of Kinnersley and Chitre[9,10] (KC) satisfies the two partial differential equations

$$\nabla F(t)=\frac{it}{S^2(t)}[(1-2tz)\nabla H-2t\rho\tilde{\nabla}H]\epsilon F(t), \quad \text{(A1)}$$

$\nabla=(\partial/\partial\rho,\partial/\partial z)$, $\tilde{\nabla}=(\partial/\partial z,-\partial/\partial\rho)$, whose compatibility is guaranteed by the differential equations for the matrix potential $H\,[=H_{AB}=H_{AB}(\rho,z)]$,[7,8]

$$\nabla H=i\rho^{-1}\mathbf{g}\epsilon\tilde{\nabla}H. \quad \text{(A2)}$$

In Eqs. (A1) and (A2) we have used the notation,

$$S(t)=[(1-2tz)^2+4t^2\rho^2]^{1/2}$$
$$=[(1-2t\beta)^2-4t^2\alpha^2]^{1/2},\quad S(0)=1, \quad \text{(A3)}$$

$$\epsilon=\begin{pmatrix}0&1\\-1&0\end{pmatrix}, \quad \text{(A4)}$$

and $\mathbf{g}=\operatorname{Re}H$ is the 2×2 block of the metric [see Eqs. (1.1)–(1.5)]. If cylindrically symmetric gravitational wave solutions rather than axially symmetric stationary solutions are under consideration, one should replace the coordinates (ρ,z) by (α,β), where $\alpha=i\rho,\beta=z$, and perform the necessary analytic continuations.

If $F_0(t)$ is a particular solution of Eq. (A1) for given $\mathbf{g}=\operatorname{Re}H$, the general solution is

$$F(t)=F_0(t)g(t), \quad \text{(A5)}$$

where $g(t)$ is a rational matrix function of t only, not of ρ or z. Since some of the freedom in $F(t)$ is used up by the initial conditions,

$$F(0)=i\epsilon,\quad \dot{F}(0)=H, \quad \text{(A6a,b)}$$

and first integrals,

$$\det F(t)=-1/S(t), \quad \text{(A7a)}$$

$$S(t)F^*(t)=2it\mathbf{g}\epsilon F(t)-(1-2tz)F(t), \quad \text{(A7b)}$$

$F^*(t)$ being the complex conjugate of $F(t^*)$, $g(t)$ is accordingly restricted to obey

$$g(0)=I,\quad \det g(t)=1,\quad g^*(t)=g(t). \quad \text{(A8)}$$

This nonuniqueness in the definition of $F(t)$ gives rise to the infinite-dimensional group of gauge transformations[6,9] acting on the hierarchy of KC potentials [coefficients in the power series expansion (1.9)] which leave invariant the metric $\mathbf{g}=\operatorname{Re}H$.

Hauser and Ernst[5] have proved that, under general circumstances, there exists a unique gauge (denoted here "special HE gauge") such that $F(t)$ is analytic in the whole complex t plane except for quadratic branch points at the two zeros of $S(t)$ joined by a cut and that $F(t)\begin{pmatrix}1&0\\0&t\end{pmatrix}$ is analytic at $t=\infty$. Furthermore, the two branch points are of index $-\frac{1}{2}$ in such a way that

$$F(t)=A(t)+B(t)/S(t), \quad \text{(A9)}$$

where $A(t)$ and $B(t)$ are analytic at the zeros of $S(t)$, but will in general have (ρ,z)-independent singularities elsewhere. The principal condition in Hauser and Ernst's theorem is that $F(t)$ be evaluated in an open (ρ,z) domain containing the origin (0, 0) for which $\mathscr{E}=H_{11}$ is an analytic function of ρ and z and $f=\operatorname{Re}\mathscr{E}\neq0$ [see Ref. 5 for a detailed discussion of the analyticity properties of $\mathscr{E}(\rho,z)$ and $F(\rho,z,t)$].

In this appendix, we wish to give an explicit determination of the matrix functions $A(t)$ and $B(t)$ in Eq. (A9), or, equivalently, a formula in terms of $F(t)$ for

$$\tilde{F}(t)=A(t)-B(t)/S(t). \quad \text{(A10)}$$

$\tilde{F}(t)$ is simply the result of analytically continuing $F(t)$ across the branch cut and onto the second Riemann sheet. This two-sheeted Riemann surface will not necessarily be the whole Riemann surface for the maximally extended function $F(t)$ because of possible (ρ,z)-independent branch points in $\tilde{F}(t)$ on the second sheet. Since the differential equation (A1) is even in $S(t)$, it follows that $\tilde{F}(t)$ also satisfies Eq. (A1) and so

$$\tilde{F}(t)=iF(t)h(t), \quad \text{(A11)}$$

where $h(t)$ is some matrix function of t only, to be determined (the imaginary unit i is introduced for later convenience). The inverse relation, $F(t)=i\tilde{F}(t)h(t)$, and Eqs. (A7a,b) give the following constraints on $h(t)$:

$$\det h(t)=1,\quad h(t)^2=-I,$$
$$\operatorname{tr}h(t)=0,\quad h^*(t)=h(t) \quad \text{(A12)}$$

[cf. Eqs. (6.53) and (6.54) of Ref. 16]. These relations hold for any choice of gauge for $F(t)$.

The determination of $h(t)$ in terms of the Ernst potential $\mathscr{E}$ invokes some of the main results of Ref. 5. First, since special HE gauge is unique when it exists,[5] it follows that, in this gauge, $h(t)$ is uniquely determined in terms of $\mathscr{E}$. (It will be seen that the converse is also true.) First, let us use the homogeneous Hilbert problem (HHP) to find how $h(t)$ transforms under an element of the Geroch group represented by $u(t)$. With a prime denoting transformed variables, the HHP reads,

$$X_-(t)=F'(t)u(t)F(t)^{-1}, \quad \text{(A13)}$$

where $X_-(t)$ is analytic in L_- (the exterior of L) and at $t=\infty$, $F'(t)$ and $F(t)$ are analytic in L_+ (the interior of L). On the second Riemann sheet, we must have

$$X_-(t)=\tilde{F}'(t)u(t)\tilde{F}(t)^{-1}$$
$$=F'(t)h'(t)u(t)h(t)^{-1}F(t)^{-1}, \quad \text{(A14)}$$

since neither $X_-(t)$ nor $u(t)$ can have branch points at the zeros of $S(t)$ in L_-. Comparison of Eq. (A14) with (A13) shows that

$$h'(t)=u(t)h(t)u(t)^{-1}. \quad \text{(A15)}$$

This derivation does not presuppose that either $F(t)$ or $F'(t)$ is in special HE gauge. If both are in the special gauge, $u(t)$ is necessarily analytic in L_- and obeys the HE boundary condition at $t=\infty$ [see Eq. (1.13d)].[5] It is obvious from Eqs. (A5) and (A11) that an arbitrary gauge change,

$$F(t)\rightarrow F(t)g(t), \quad \text{(A16a)}$$

has the effect

$$h(t)\rightarrow g(t)^{-1}h(t)g(t). \quad \text{(A16b)}$$

The next step is to find all $u(t)$ analytic in L_- which map flat space $\mathscr{E}=1$ to a given stationary solution

$\mathscr{E}' = \mathscr{E}'(\rho, z)$, preserving special HE gauge. It will be necessary that $\mathscr{E}'$ be analytic in a (ρ, z) domain containing the origin (0, 0). The following formula of Hauser and Ernst[5] gives implicitly all group elements $u(t)$ which map any given initial solution $\mathscr{E}$ to any given final solution $\mathscr{E}'$ preserving special HE gauge:

$$(t\mathscr{E}', i)u(t)\begin{pmatrix} -it^{-1} \\ \mathscr{E} \end{pmatrix} = 0, \tag{A17}$$

where $\mathscr{E}$ and $\mathscr{E}'$ are to be evaluated at $\rho = 0, z = (2t)^{-1}$. The real and imaginary parts of this equation, together with the determinant condition (1.13a), provide three equations for the four components of $u(t)$. Now put $\mathscr{E} = 1$ and

$$\mathscr{E}' = f + i\psi = f(0, (2t)^{-1}) + i\psi(0, (2t)^{-1}) \tag{A18}$$

in Eq. (A17) and solve for the components of $u(t)$ in terms of f, ψ, and one arbitrary function of t (hereafter we drop the prime on $\mathscr{E}'$). The result is

$$u(t) = \begin{pmatrix} f^{-1/2} & 0 \\ -t\psi f^{-1/2} & f^{1/2} \end{pmatrix}\begin{pmatrix} \cos\theta(t) & t^{-1}\sin\theta(t) \\ -t\sin\theta(t) & \cos\theta(t) \end{pmatrix}, \tag{A19}$$

where $\theta(t)$ is analytic in L_- and at $t = \infty$, but otherwise arbitrary. The matrix on the right in Eq. (A19) represents the general **B**-group element[10] which maps flat space to itself, preserving special HE gauge. This result also shows incidentally how $\mathscr{E}(\rho, z)$ can be deduced uniquely in terms of $\mathscr{E}(0, z)$ from the HHP.

Finally, in order to explicitly calculate $h(t)$ for any given solution $\mathscr{E}$, one need only introduce the flat space expression in the right-hand side of Eq. (A15). For flat space,[9]

$$F(t) = \begin{pmatrix} \dfrac{t}{S(t)} & \dfrac{i}{S(t)} \\ -\dfrac{it\nu(t)}{S(t)} & \dfrac{\mu(t)}{S(t)} \end{pmatrix}, \tag{A20}$$

where

$$\mu(t) = \frac{1 - 2tz - S(t)}{2t}, \quad \nu(t) = \frac{1 - 2tz + S(t)}{2t}. \tag{A21}$$

It follows that $h(t)$ for flat space is given by

$$h(t) = \begin{pmatrix} 0 & -t^{-1} \\ t & 0 \end{pmatrix}. \tag{A22}$$

Hence, substituting Eqs. (A19) and (A22) into Eq. (A15), we find that $h(t)$ for any solution $\mathscr{E} = f + i\psi$ in special HE gauge is given by

$$h(t) = \begin{pmatrix} -\psi f^{-1} & -t^{-1}f^{-1} \\ t(f + \psi^2 f^{-1}) & \psi f^{-1} \end{pmatrix}, \tag{A23}$$

with f and ψ evaluated at $\rho = 0, z = (2t)^{-1}$. Notice that the HE boundary condition that

$$F(t)\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$$

be analytic at $t = \infty$ is satisfied on both Riemann sheets. Notice also that if $h(t)$ is known, then $\mathscr{E}(0, z)$ can be deduced immediately from Eq. (A23), and then $\mathscr{E}(\rho, z)$ and $F(t)$ are determined uniquely by the HHP.

Although the $F(t)$ potential for the general Weyl solution, $\mathscr{E} = e^{2\chi}$, $\chi_{\rho\rho} + \rho^{-1}\chi_\rho + \chi_{zz} = 0$, is well known, it is not at all obvious how to identify which gauge is special HE gauge. The expression given in Ref. 12 can be written

$$F(t) = \begin{pmatrix} e^{\chi} & 0 \\ 0 & e^{-\chi} \end{pmatrix}\begin{pmatrix} \dfrac{t}{S(t)} & \dfrac{i}{S(t)} \\ -\dfrac{it\nu(t)}{S(t)} & \dfrac{\mu(t)}{S(t)} \end{pmatrix}\begin{pmatrix} e^{\beta(t)} & 0 \\ 0 & e^{-\beta(t)} \end{pmatrix}, \tag{A24}$$

where $\beta(t) = \beta(\rho, z, t)$ satisfies

$$\nabla\beta(t) = \frac{(1 - 2tz)\nabla\chi - 2t\rho\tilde{\nabla}\chi}{S(t)}, \quad \beta(0) = \chi. \tag{A25}$$

$\beta(t)$ is defined up to an arbitrary additive function of t only. This arbitrariness corresponds to the freedom to change gauge in $F(t)$ with a diagonal $g(t)$ matrix. Since $\nabla\beta(t)$ is odd in $S(t)$, there is a natural particular integral of Eq. (A25), which we denote $\beta_{\rm odd}(t)$, which is also odd in $S(t)$. This natural gauge for the Weyl $F(t)$-potential is precisely the gauge used by KC to derive the Kerr and generalized Tomimatsu–Sato solutions from special Weyl solutions.[10] In Ref. 16, we calculated $\tilde{F}(t)$ with the choice $\beta(t) = \beta_{\rm odd}(t)$ and found that $h(t)$ is the same for all Weyl solutions and is given by the flat space expression (A22). However, this natural gauge is *not* special HE gauge (except for flat space) as $\beta_{\rm odd}(t)$ has (ρ, z)-independent singularities in the complex t plane. A direct calculation shows that if

$$\beta(t) = \beta_{\rm odd}(t) + \delta(t) \tag{A26}$$

in Eq. (A24), where $\delta(t)$ is some function of t only, then

$$h(t) = \begin{pmatrix} 0 & -t^{-1}e^{-2\delta(t)} \\ te^{2\delta(t)} & 0 \end{pmatrix}. \tag{A27}$$

If $F(t)$ is in special HE gauge, then $e^{\pm\beta(t)}$ must be analytic throughout the t plane (including $t = \infty$) except for quadratic branch points at the zeros of $S(t)$. In that case, $\delta(t)$ is uniquely determined by equating the right-hand sides of Eqs. (A27) and (A23) with $f = e^{2\chi}$, $\psi = 0$. The result is

$$\delta(t) = \chi(0, (2t)^{-1}). \tag{A28}$$

A direct proof of this result can be obtained from the contour integral expression (3.47b) by observing that the residue of the integrand at $t' = t$ is even in $S(t)$ and the remainder of the integral is odd in $S(t)$ and that $\xi(t) = -\chi(0, (2t)^{-1})$.

APPENDIX B: RELATIONSHIP BETWEEN $\psi(\lambda)$ AND $\psi(\alpha^2/\lambda)$

The complex spectral parameter λ used by Belinskii and Zakharov[2] (BZ) is related to t by the quadratic transformation,[16]

$$t = t(\lambda) = \lambda/(\lambda^2 + 2\beta\lambda + \alpha^2), \tag{B1}$$

so that the image of the full λ plane is the two-sheeted Riemann t surface which is the domain of $S(t)$ with branch points at the two zeros,

$$t = \tfrac{1}{2}(\beta \pm \alpha)^{-1}, \tag{B2}$$

of $S(t)$. To fix ideas, we consider (α, β) domains for which $|\beta| > \alpha > 0$. Then the branch cut may be taken as the finite segment of the real axis joining the zeros and not passing through $t=0$. For t on the first Riemann sheet (excluding cut, including $t=\infty$), the inverse of Eq. (B1) is

$$\lambda=\mu(t)=[1-2t\beta-S(t)]/2t, \tag{B3a}$$

and the image in the λ plane is the open disk, $|\lambda|<\alpha$, denoted Γ_2 (note $t=0$ implies $\lambda=0$). For t on the second Riemann sheet (excluding cut, including $t=\infty$), the inverse of Eq. (B1) is

$$\lambda=\nu(t)=[1-2t\beta+S(t)]/2t=\alpha^2/\mu(t), \tag{B3b}$$

and the image in the λ plane is the region, $|\lambda|>\alpha$, denoted Γ_1, as well as $\lambda=\infty$. The two zeros of $\lambda^2+2\beta\lambda+\alpha^2$, namely $\lambda=\lambda_2$ in Γ_2 and $\lambda=\lambda_1$ in Γ_1, map to the two points at infinity of the Riemann t surface. The circle $|\lambda|=\alpha$ (denoted Γ) maps one-to-two onto the branch cut itself. To understand this a little better, consider counterclockwise circles, $|\lambda|=\alpha-\delta$ in Γ_2 and $|\lambda|=\alpha+\delta$ in Γ_1, where δ is small and positive. The former maps to a clockwise closed curve enclosing and closely fitting the cut in the t plane, the latter to a similar but counterclockwise curve on the second Riemann sheet.

The BZ matrix eigenfunction $\psi(\lambda)$ is related to $F(t)$ by[16]

$$\psi(\lambda)=t^{-1}S(t)P(t), \quad |\lambda|<\alpha, \tag{B4}$$

where the matrix functions $P(t)$ and $Q(t)$ are defined by

$$F(t)=P(t)+iQ(t), \quad F^*(t)=P(t)-iQ(t). \tag{B5}$$

This result was proved in Ref. 16 by directly comparing the differential equations for $\psi(\lambda)$ and $P(t)$. The differential equations also allow a multiplicative matrix function of t only (reducing to the unit matrix at $t=0$) on the right of Eq. (B4) (gauge change + t-dependent rescaling) but without loss of generality we set it to be identically unity in order that $\psi(\lambda)$ be analytic in Γ_2 whenever $F(t)$ is in special HE gauge. The HE condition (1.11) at $t=\infty$ also implies corresponding conditions on $\psi(\lambda)$ at the two zeros of $\lambda^2+2\beta\lambda+\alpha^2$ [see Eq. (2.4)].

Analytic continuation of Eq. (B4) to $|\lambda|>\alpha$ gives, with an obvious notation,

$$\psi(\lambda)=-t^{-1}S(t)\tilde{P}(t), \quad |\lambda|>\alpha, \tag{B6a}$$

or, equivalently,

$$\psi(\alpha^2/\lambda)=-t^{-1}S(t)\tilde{P}(t), \quad |\lambda|<\alpha. \tag{B6b}$$

[In Eq. (B6a), $\lambda=\nu(t)$; in Eq. (B6b), $\lambda=\mu(t)$, $\alpha^2/\lambda=\nu(t)$.] Now, from Eqs. (B5) and (A11),

$$\tilde{P}(t)=-Q(t)h(t), \quad \tilde{Q}(t)=P(t)h(t), \tag{B7a,b}$$

where $h(t)$ is given by Eq. (A23) if $F(t)$ is in special HE gauge, and from Eqs. (A16a,b) otherwise. We reproduce here some simple identities given in Ref. 16 which are immediate consequences of Eqs. (A7a,b) here:

$$P(t)=-\nu^{-1}(t)\mathbf{g}\boldsymbol{\epsilon}Q(t), \quad Q(t)=\mu^{-1}(t)\mathbf{g}\boldsymbol{\epsilon}P(t), \tag{B8a,b}$$

$$\det P(t)=t\mu(t)/S^2(t), \quad \det Q(t)=t\nu(t)/S^2(t), \tag{B9a,b}$$

$$\det\psi(\lambda)=\lambda^2+2\beta\lambda+\alpha^2. \tag{B10}$$

From Eqs. (B7a) and (B8b), we have

$$\tilde{P}(t)=-\mu^{-1}(t)\mathbf{g}\boldsymbol{\epsilon}P(t)h(t). \tag{B11}$$

Finally, from Eqs. (B4) and (B6b), we obtain the desired relationship:

$$\psi(\alpha^2/\lambda)=\lambda^{-1}\mathbf{g}\boldsymbol{\epsilon}\psi(\lambda)h(t). \tag{B12}$$

Although Eq. (B12) was derived for $|\lambda|<\alpha$, it holds for all λ. This can be seen from the inverse relation

$$\begin{aligned}\psi(\lambda)&=-\lambda\boldsymbol{\epsilon}\mathbf{g}^{-1}\psi(\alpha^2/\lambda)h(t)^{-1}\\&=(\lambda/\alpha^2)\mathbf{g}\boldsymbol{\epsilon}\psi(\alpha^2/\lambda)h(t),\end{aligned} \tag{B13}$$

where we have used $\boldsymbol{\epsilon}^{-1}=-\boldsymbol{\epsilon}$, $\det\mathbf{g}=\alpha^2$, $h(t)^-=-h(t)$, the symmetry of $\mathbf{g}$, and the following identity satisfied by any nondegenerate matrix M:

$$\boldsymbol{\epsilon}M^T\boldsymbol{\epsilon}=-(\det M)M^{-1}, \tag{B14}$$

^{T} denoting matrix transpose.

[1]See for example, M. J. Ablowitz, Stud. Appl. Math. **58**, 17 (1978); M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Stud. Appl. Math. **53**, 249 (1974); R. Bullough and P. J. Caudrey (editors), *Solitons* (Springer-Verlag, Berlin, 1980); J. A. DeSanto, A. W. Sáenz, and W. W. Zachary (editors), *Mathematical Methods and Applications of Scattering Theory*, Lecture Notes in Physics **130** (Springer-Verlag, Berlin, 1980), p. 233ff; R. Hermann, *The Geometry of Nonlinear Differential Equations, Bäcklund Transformations, and Solitons* (Math. Sci. Press, Brookline, MA, 1976); R. M. Miura (editor), *Bäcklund Transformations, the Inverse Scattering Method, Solitons, and their Applications*, Lecture Notes in Mathematics **515** (Springer-Verlag, Berlin, 1976); V. E. Zakharov and A. B. Shabat, Functional Anal. Appl. **8**, (3), 226 (1974); **13**, (3), 166 (1979).

[2]V. A. Belinskii and V. E. Zakharov, Sov. Phys. JETP **48**, 985 (1978).

[3]I. Hauser and F. J. Ernst, Phys. Rev. D **20**, 362 (1979) (vacuum); Phys. Rev. D **20**, 1783 (1979) (electrovac).

[4]I. Hauser and F. J. Ernst, J. Math. Phys. **21**, 1126 (1980) (vacuum); J. Math. Phys. **21**, 1418 (1980) (electrovac).

[5]I. Hauser and F. J. Ernst, J. Math. Phys. **22**, 1051 (1981).

[6]R. Geroch, J. Math. Phys. **13**, 394 (1972).

[7]W. Kinnersley, J. Math. Phys. **18**, 1529 (1977).

[8]W. Kinnersley and D. M. Chitre, J. Math. Phys. **18**, 1538 (1977).

[9]W. Kinnersley and D. M. Chitre, J. Math. Phys. **19**, 1926 (1978).

[10]W. Kinnersley and D. M. Chitre, Phys. Rev. Lett. **40**, 1608 (1978); J. Math. Phys. **19**, 2037 (1978).

[11]C. M. Cosgrove, J. Math. Phys. **22**, 2624 (1981).

[12]C. Hoenselaers, W. Kinnersley, and B. C. Xanthopoulos, Phys. Rev. Lett. **42**, 481 (1979); J. Math. Phys. **20**, 2530 (1979); B. C. Xanthopoulos, *ibid.* **22**, 1254 (1981).

[13]B. K. Harrison, Phys. Rev. Lett. **41**, 1197 (1978); Phys. Rev. D **21**, 1695 (1980).

[14]C. M. Cosgrove, in Proceedings of the Second Marcel Grossmann Meeting on the Recent Developments of General Relativity, Trieste, Italy, 1979 (to be published).

[15]V. A. Belinskii and V. E. Zakharov, Sov. Phys. JETP **50**, 1 (1979); G. A. Alekseev and V. A. Belinskii, *ibid.* **51**, 655 (1980); V. A. Belinskii, JETP Lett. **30**, 28 (1979); G. A. Alekseev, *ibid.* **32**, 277 (1980); A. Tomimatsu, Prog. Theor. Phys. (Letters) **63**, 1054 (1980).

[16]C. M. Cosgrove, J. Math. Phys. **21**, 2417 (1980).

[17]An example of an invariance group larger than **K** is the infinite-dimensional group formed by iterating Neugebauer's Bäcklund transformations I_1 and I_2[14,16,28] until closure. All elements of this group can be factorized uniquely in the forms qg_1 and g_2q, where $g_1, g_2 \in \mathbf{K}$ and $q\in\mathbf{Q}$, the three-dimensional **Q** group[14,16,40] being closely related to I_1.

[18]D. Kramer and G. Neugebauer, Commun. Math. Phys. **10**, 132 (1968), Eq. (13).

[19]F. J. Ernst, Phys. Rev. **167**, 1175 (1968); **168**, 1415 (1968); J. Math. Phys. **15**, 1409 (1974).

[20]H. Weyl, Ann. Phys. (Leipzig) **54**, 117 (1917) (static case); T. Lewis, Proc. R. Soc. London Ser. A **136**, 176 (1932); A. Papapetrou, Ann. Phys. (Leipzig) **12**, 309 (1953).

[21]A. Einstein and N. J. Rosen, J. Franklin Inst. **223**, 43 (1937) (diagonal metric); A. S. Kompaneets, Sov. Phys. JETP **7**, 659 (1958); A. Papapetrou, Ann. Inst. H. Poincaré A **4**, 83 (1966).

[22]P. Szekeres, Nature **228**, 1183 (1970); J. Math. Phys. **13**, 286 (1972); K. A. Kahn and R. Penrose, Nature **229**, 185 (1971).

[23]W. B. Bonnor, Z. Phys. **161**, 439 (1961); R. Catennaci and J. Diaz Alonso, J. Math. Phys. **17**, 2232 (1976).
[24]P Forgács, Z. Horváth, and L. Palla, Phys. Rev. Lett. **45**, 505 (1980).
[25]E. Kyriakopoulos, Gen. Rel. Grav. **12**, 989 (1980).
[26]V. E. Zakharov and A. V. Mikhailov, Sov. Phys. JETP **47**, 1017 (1978).
[27]In order to convert a real-valued solution of an elliptic equation (e.g., stationary axisymmetric) to a physically realistic solution of a hyperbolic equation (e.g., cylindrical wave) by the obvious complex substitution [Eq. (1.2)], it must be possible to analytically continue the parameters so that the wave solution is real-valued, and vice versa. For example, the Kerr solution maps to a double soliton wave (Ref. 2), whereas Papapetrou's wave solution (third paper of Ref. 21) does not map to a real-valued stationary solution. Also, the boundary condition of asymptotic flatness applicable to stationary gravitational or electrovac solutions may not be the desired boundary condition in some of the other physical theories.
[28]G. Neugebauer, J. Phys. A **12**, L67 (1979); **13**, L19 (1980); J. Phys. A **13**, 1737 (1980).
[29]Our convention for the position of the entries of the matrix $F(t)$ is in accordance with the SL(2)-tensor $F_{AB}(t)$ of Refs. 9–12 and 16. The HE matrices, $F(t)$, $u(t)$, etc., can be obtained from ours by interchanging the rows (with each other) and columns (with each other).
[30]Condition (1.11) requires that $\omega = 0$ [see Eq. (1.4)] on the z axis near (0,0). If $\omega(0,0)\neq 0$, then the theorem holds except that condition (1.11) should be replaced by the weaker condition that $F(t)$ only be analytic at $t=\infty$, and the gauge is unique up to one arbitrary constant. This slightly more general gauge is called "modified special HE gauge" in Ref. 11. Equation (A2) and the analyticity of $\mathscr{E}(\rho,z)$ imply that $\omega(0,z)$ is constant, say b, and so condition (1.11) can be brought about by a coordinate transformation, $x'^1 = x^1 - bx^2$, $x'^2 = x^2$, in the metric (1.1) or (1.3). The effect on the $F(t)$ potential is given by

$$F'(t) = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} F(t) \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

and the representing matrix is

$$u(t) = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}.$$

When modified special HE gauge is under consideration, condition (1.13d) on $u(t)$ should be replaced by analyticity of $u(t)$ at $t=\infty$.
[31]The variable W of Ref. 2 is given by $W = (2t)^{-1}$.
[32]In contrast to Refs. 2 and 26, we allow $G_0(\lambda)\equiv G_0(t)$ to admit analytic continuation off the contour Γ.
[33]The HKX transformation (3.20) is said to be "null" because $q^{AB}\equiv q^A q^B$ is a null tensor or, equivalently, a degenerate matrix, and "generalized" because the original HKX transformation of Ref. 12 refers to the special case $q^1 = 1$, $q^2 = 0$.
[34]D. Zipoy, J. Math. Phys. **7**, 1137 (1966); B. Voorhees, Phys. Rev. D **2**, 2119 (1970); P. Y. Chou, Am. J. Math. **53**, 289 (1931). Special cases are as follows: Schwarzschild, $\delta = 1$; Darmois, $\delta = 2$; Curzon, $\delta = \infty$.
[35]A. Tomimatsu and H. Sato, Prog. Theor. Phys. **50**, 95 (1973); M. Yamazaki, Prog. Theor. Phys. **57**, 1951 (1977); S. Hori, Prog. Theor. Phys. **59**, 1870 (1978); C. M. Cosgrove, J. Phys. A **10**, 1481 (1977); P. Dale, Proc. R. Soc. London Ser. A **362**, 463 (1978).
[36]D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (VEB Deutscher Verlag der Wissenschaften, Berlin, 1980), Secs. 30.4, 30.5.
[37]D. Kramer and G. Neugebauer, Phys. Lett. A **75**, 259 (1980).
[38]The extended null HKX transformation of rank $N-1$ is a special BZ $2N$-soliton transformation for which $\chi_2(\lambda)$ has just one pole of multiplicity $2N$ at $\lambda = \mu(s)$ in Γ_2 and only two of the $2N$ integration constants are independent. Allowing all $2N$ integration constants to be arbitrary yields the combined HKX transformation of ranks, $0,1,\ldots,N-1$. The infinitesimal versions of these transformations are infinite linear combinations of $q^X q^Y \gamma^{(k)}_{XY}$, $k = 0, 1, 2, \ldots$, q^X fixed. Reference 12 treats the subclass for which $q^1 = 1$, $q^2 = 0$.
[39]Equations (7.16) and (7.17) of Ref. 16 give the transforms of $G_{AB}(t_1,t_2)$ and $F_{AB}(t)$, respectively, under the $s=\infty$ nonnull HKX transformation. Due to a printing error, the first part of Eq. (7.16) was omitted. The omitted part should read

$$G'_{AB}(t_1,t_2) = G_{AB}(t_1,t_2) + \cdots.$$

[40]C. M. Cosgrove, in *Gravitational Radiation, Collapsed Objects, and Exact Solutions*, Proceedings of the Einstein Centenary Summer School, edited by C. Edwards, Lecture Notes in Physics **124** (Springer–Verlag, Berlin, 1980), pp. 444–453.